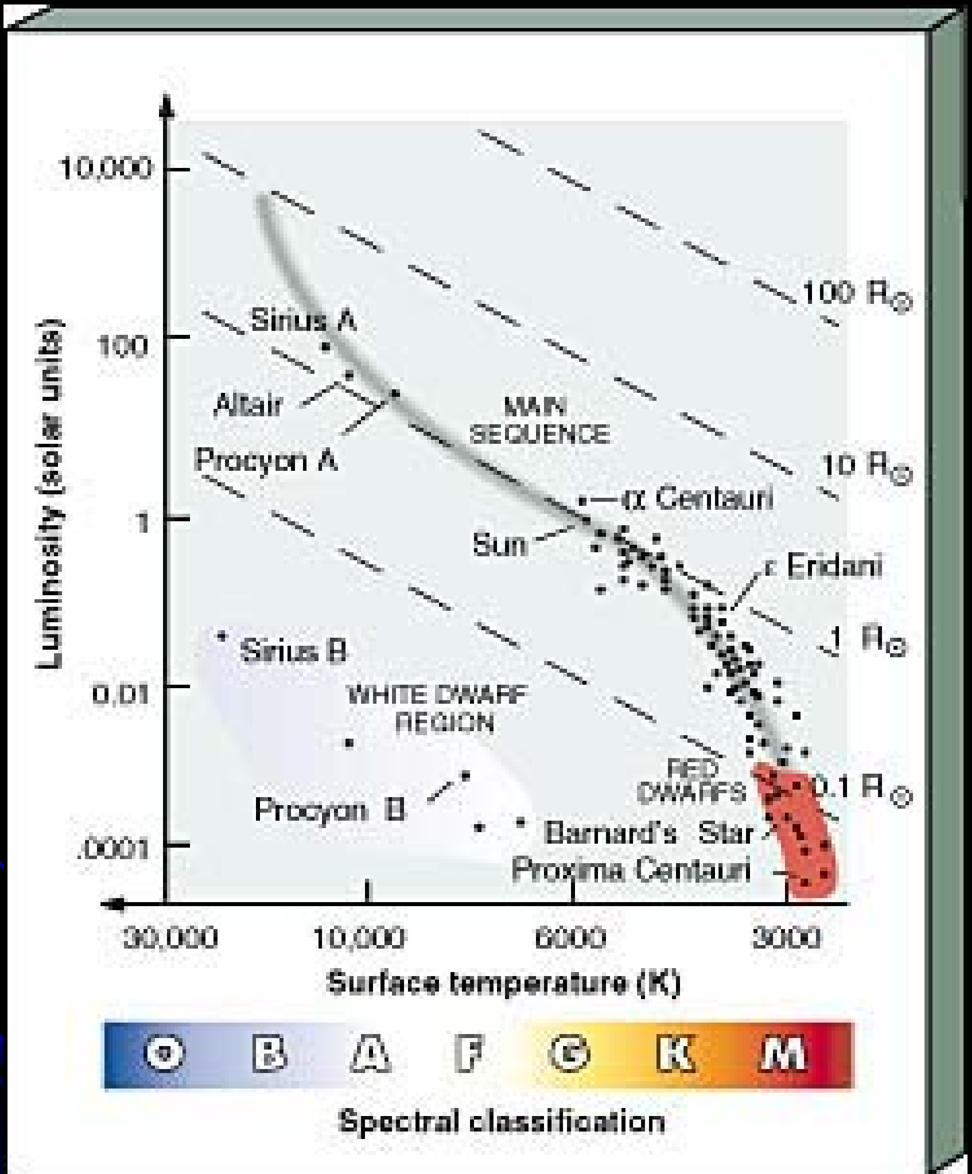
Relativistic Universe

Quantitative theory of star structure and star evolution

The Hertzprung-Russel diagram



Three main stages

- There are three main stages of star evolution:

1. From molecular cloud to main sequence star;
2. Main sequence;
3. From main sequence, through red giant to white dwarf (neutron star, black hole).

Both evolution and internal structure of stars are now well understood!

The interior of a star …

- Can be described by Euler equation of fluid dynamics

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\nabla\Phi - \frac{1}{\rho}\nabla P$$

Euler equation

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\nabla\Phi - \frac{1}{\rho}\nabla P$$

u is vector of velocity of small element of the fluid;

ρ is density;

P is pressure;

Φ is Newtonian potential.

In equilibrium

- Equation of hydrostatic equilibrium

$$\nabla P = -\rho \nabla \Phi$$

Stars in equilibrium

- The equilibrium equation implies relation among pressure, mass, and radius.

$$\nabla P = -\rho \nabla \Phi \quad \Rightarrow$$

$$\frac{dP}{dr} \sim \frac{M(r)\rho}{r^2} \sim \frac{M\left(M/R^3\right)}{R^2}$$

- On the other hand

$$\frac{dP}{dr} \sim \frac{P}{R} \quad \Rightarrow P \sim \frac{M^2}{R^4}$$

For white dwarf

- The star is composed of degenerate gas of electrons, for which

$$P \sim \rho^{5/3} \sim \frac{M^{5/3}}{R^5}$$

$$\frac{dP}{dr} \sim \frac{P}{R} \quad \Rightarrow P \sim \frac{M^2}{R^4}$$

It follows that

$$M \sim R^{-3}$$

- As the mass of configuration increases the radius decreases. More moassive the white dwarf, the smaller its radius. Adding matter to white dwarf (by accretion, for example) causes its radius to decrease.

- Sooner or later the equation of state must change over to the fully relativistic one. Here

$$P \sim \rho^{4/3} \sim \frac{M^{4/3}}{R^4} \quad \Rightarrow \quad M \sim const$$

- Thus for fully relativistic degenerate gas, there is a unique mass for which the configuration is stable. If this mass is exceeded, the star would collapse. Thus white dwarfs cannot be more massive than this limiting mass, called the Chandrasekhar limit and equal 1.4 $M_{\odot}$

Let us derive a time averaged form of Euler equation

- By standard calculation we can convince ourselves that

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = \frac{d\vec{u}}{dt}$$

- So that we can write

$$\rho \frac{d\vec{u}}{dt} + \rho \nabla \Phi + \nabla P = 0$$

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- We multiply by $\vec{r}$ and then integrate over volume of the star

$$\int_V \rho \vec{r} \cdot \frac{d\vec{u}}{dt} dV + \int_V \rho \vec{r} \cdot \nabla \Phi \, dV + \int_V \vec{r} \cdot \nabla P \, dV = 0$$

Consider this equation term by term

$$\int_V \rho \vec{r} \cdot \frac{d\vec{u}}{dt} dV =$$

$$\int_V \rho \vec{r} \cdot \frac{d^2\vec{r}}{dt^2} dV =$$

$$\frac{1}{2}\frac{d^2}{dt^2}\int_V \rho \vec{r}^{\,2}\, dV - \int_V \rho \vec{u}^{\,2}\, dV = \frac{1}{2}\frac{d^2 I}{dt^2} - 2T$$

I is the total moment of inertia about the origin,
T is total kinetic energy

$$\int_V \rho \vec{r} \cdot \nabla P \, dV =$$

$$\int_S \vec{r} \cdot \vec{n} \, P_s \, dA - 3\int_V P dV = \qquad (\nabla \vec{r} \equiv 3)$$

$$-3\int_V P \, dV = -2U$$

The pressure *P* vanishes on the surface;
We assume that gas inside the star is ideal.

$$\int_V \rho \vec{r} \cdot \nabla \Phi \, dV = \Omega$$

For $1/r^2$ force this is just the total potential energy.

Non-averaged virial theorem

- Finally we get

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2(T + U) + \Omega$$

Virial theorem

- If we consider a system in equilibrium, or at least long-term steady state, then the time average of moment of inertia vanishes and we have

$$2\langle T\rangle + 2\langle U\rangle + \langle \Omega\rangle = 0$$

Consequences

- Let us estimate the parameters of the cloud from which star can eventually form. The internal kinetic energy of the gas in the cloud must be less than one half the gravitational energy, in order for moment of inertia to show any accelerative contraction.

- For uniform gas confined to a sphere with radius R_c and of temperature T

$$2\left(\frac{3\rho kT}{2\mu m_h}\right)\left(\frac{4\pi R_c^{\,2}}{3}\right) \leq \frac{GM^2}{R_c}$$

μm_h is an effective mass of a particle in the gas.

$$R_c \leq \frac{GM\mu m_h}{3kT} \approx \frac{0.25(M/M_\odot)}{T}\ pc$$

- This distance is called Jeans length, it is the distance below which a gas cloud becomes gravitationally unstable.
- For a solar mass of material at the typical temperature of 50K, the cloud would be smaller than about 5×10^{-3} pc, with mean density greater than about 10^8 particles per cubic cm.
- These are not typical conditions in the interstellar cloud!

The parameter μ

- It is convenient to devide teh composition of the stellar matter into three categories:
- **X** – mass fraction of gas which is hydrogen
- **Y** – mass fraction of gas which is helium
- **Z** – mass fraction of gas which is everything else (metals)

- Now we want to calculate the number of particles in the unit volume of ionized gas.
- Hydrogens contributes 2**X** : (electron + proton)
- Helium contributes ¾ **Y**: (2 electrons + α particle but the mass is 4 times that of hydrogen)
- Metals contribute ½ **Z**: (z electrons + 1 nuclei, but the mass is 2z that of hydrogen)

- All together we have

$$N = \frac{\rho}{m_h}\left(2\mathbf{X} + {}^{3}\!/_{4}\,\mathbf{Y} + {}^{1}\!/_{2}\,\mathbf{Z}\right)$$

- But **X** + **Y** + **Z** =1

$$N = \frac{1}{2}\frac{\rho}{m_h}\left(3\mathbf{X} + {}^{1}\!/_{2}\,\mathbf{Y} + 1\right)$$

- For ideal gas

$$P = NkT = \frac{\rho k T}{\mu m_h}$$

- where

$$\mu = \frac{2}{1 + 3\mathbf{X} + \frac{1}{2}\mathbf{Y}}$$

Interior of a star

- It is assumed that the density is a monotonically decreasing function of radius

$$\rho(r) \le \langle \rho \rangle (r) \quad \text{for} \quad r > 0$$

$$\langle \rho \rangle (r) = \frac{M(r)}{4\pi r^3 / 3}, \quad M(r) = \int_0^r 4\pi \, \rho(r) \, r^2 dr$$

Poisson's equation

$$\nabla^2 \Omega = 4\pi G \rho$$

- In spherical coordinates

$$\frac{d}{dr}\left(r^2 \frac{d\Omega}{dr} \right) = 4\pi G \rho r^2$$

- Integrating

$$\frac{d\Omega}{dr} = \frac{G}{r^2}\int_0^r 4\pi r^2 \rho\, dr = \frac{GM(r)}{r^2}$$

But $\nabla P = -\rho\,\nabla\Omega \quad \Rightarrow \quad \frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$

This is the equation of hydrostatic equilibrium for spherical stars

- We introduced total mass in the interior to r by using mass conservation

$$M(r) = \int_0^r 4\pi r^2 \rho(r) dr \quad \Leftrightarrow$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r), M(0) = 0$$

Chandrasekhar variables

$$I_{\sigma,\nu}(r) \equiv \frac{G}{4\pi}\int_0^r \frac{\left[M(r)\right]^\sigma}{r^\nu}\,dM(r)$$

- For example, total gravitational energy

$$4\pi I_{1,1}(R) = G\int_0^R M(r)\rho 4\pi r^2 \frac{dr}{r}$$

- Chandrasekhar variables can be used to express some useful quantieties:

$$\langle P \rangle \equiv \int_0^M P(r) \frac{dM(r)}{M} = \frac{I_{2,4}(R)}{M};$$

$$\langle T \rangle \equiv \int_0^M T(r) \frac{dM(r)}{M} = \frac{4\pi\mu m_h}{3k} \frac{I_{1,1}(R)}{M};$$

$$\langle g \rangle \equiv \int_0^M g(r) \frac{dM(r)}{M} = \frac{I_{1,2}(R)}{M}$$

It can be checked that

$$\frac{G}{4\pi}\left[\frac{2\pi}{3}\right]^{\nu/3} \rho_c^{\ \nu/3} \frac{M^{\sigma+1-\nu/3}(r)}{\sigma+1-\nu/3} \geq$$

$$I_{\sigma,\nu}(r) \geq$$

$$\frac{G}{4\pi}\left[\frac{2\pi}{3}\right]^{\nu/3} \langle\rho\rangle^{\nu/3} \frac{M^{\sigma+1-\nu/3}(r)}{\sigma+1-\nu/3}$$

It follows that

$$\langle P \rangle \geq \frac{3GM^2}{20\pi R^4} = 5.4\times10^8 \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R_\odot}{R}\right)^4 \quad atm;$$

$$\langle T \rangle \geq \frac{G\mu m_h M}{5kR} = 4.61\times10^6\ \mu \left(\frac{M}{M_\odot}\right)\left(\frac{R_\odot}{R}\right) \quad K;$$

$$\langle g \rangle \equiv \frac{3GM}{4R^2} = 2.05\times10^2 \left(\frac{M}{M_\odot}\right)\left(\frac{R_\odot}{R}\right)^2 \quad m/s^2$$