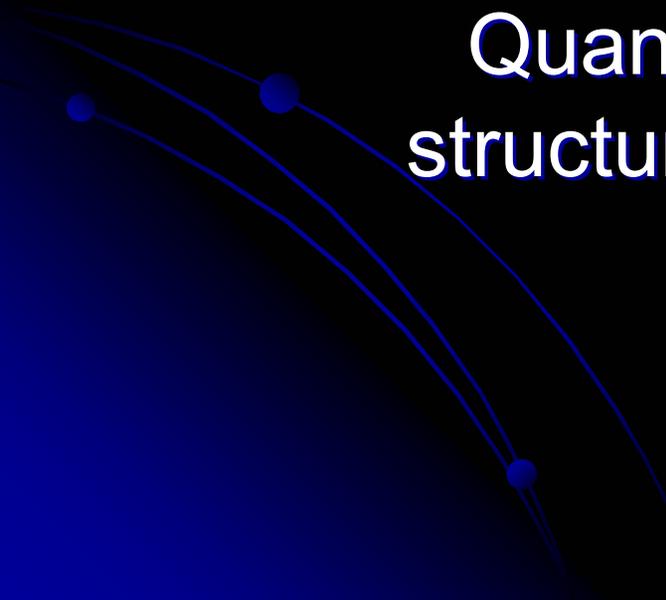


Relativistic Universe

Quantitative theory of star
structure and star evolution II



Polytropes

- To complete the theory of processes taking place in stars we still need equation of state. It turned out that significant insight into the structure and evolution of stars is provided by assuming the polytropic equation of state.

$$P(r) = K \rho(r)^{(n+1)/n}$$

- Using equation of state we can eliminate pressure from the equation of hydrostatic equilibrium for spherical stars

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$


- We differentiate this equation and use

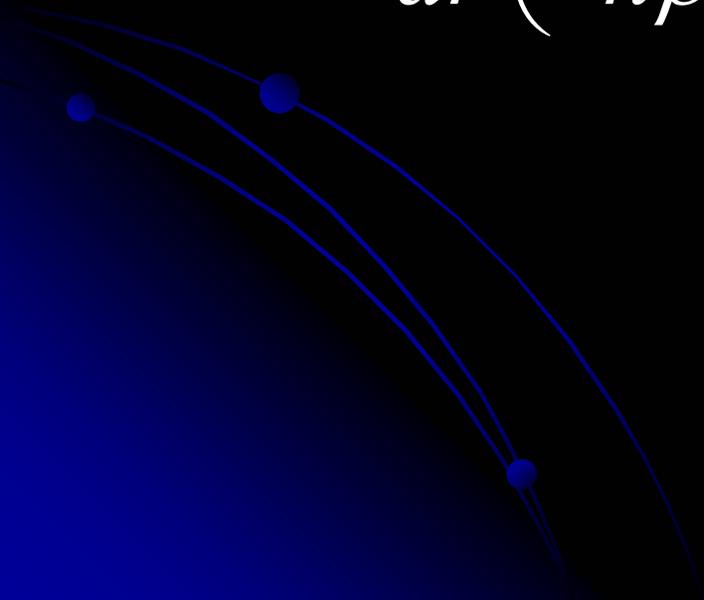
$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

$$\frac{d}{dr} \left(\frac{r^2}{\rho(r)} \frac{dP}{dr} \right) = -\frac{d}{dr} GM(r) = -4\pi G r^2 \rho(r)$$

$$\frac{dP}{dr} = K \frac{n+1}{n} \rho^{1/n} \frac{d\rho}{dr}$$

Polytropic star (Lame-Emden) equation

$$\frac{d}{dr} \left(\frac{Kr^2(n+1)}{n\rho^{(n-1)/n}} \frac{d\rho}{dr} \right) = -4\pi Gr^2 \rho;$$



Initial conditions

- We want to use natural initial conditions $\rho(0) = \rho_c$; $\rho(R) = 0$. But this naive choice does not work because it follows from the equilibrium condition that $0 = P(0) \sim \rho'(0)$. Thus the initial conditions are $\rho(0) = \rho_c$, $\rho'(0) = 0$, while radius of the star R is to be computed from $\rho(R) = 0$.

Solutions

- Polytropic equation can be solved analytically only for few (not particularly interesting) values of n . It can be, however, easily solved numerically for any n .
- For all polytropes

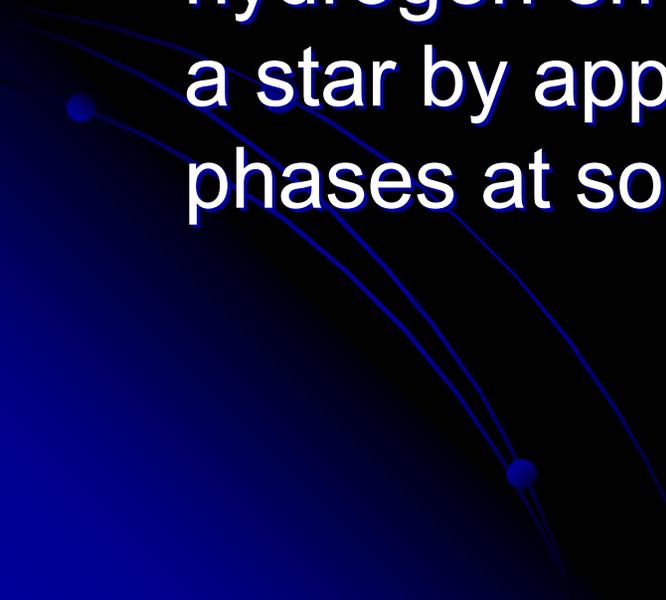
$$M^{(n-1)/n} R^{(3-n)/n} = \text{const}(n)$$

- For example, for isothermal star ($n=4$) we have

$$M \approx R^{1/3}$$



Realistic stars

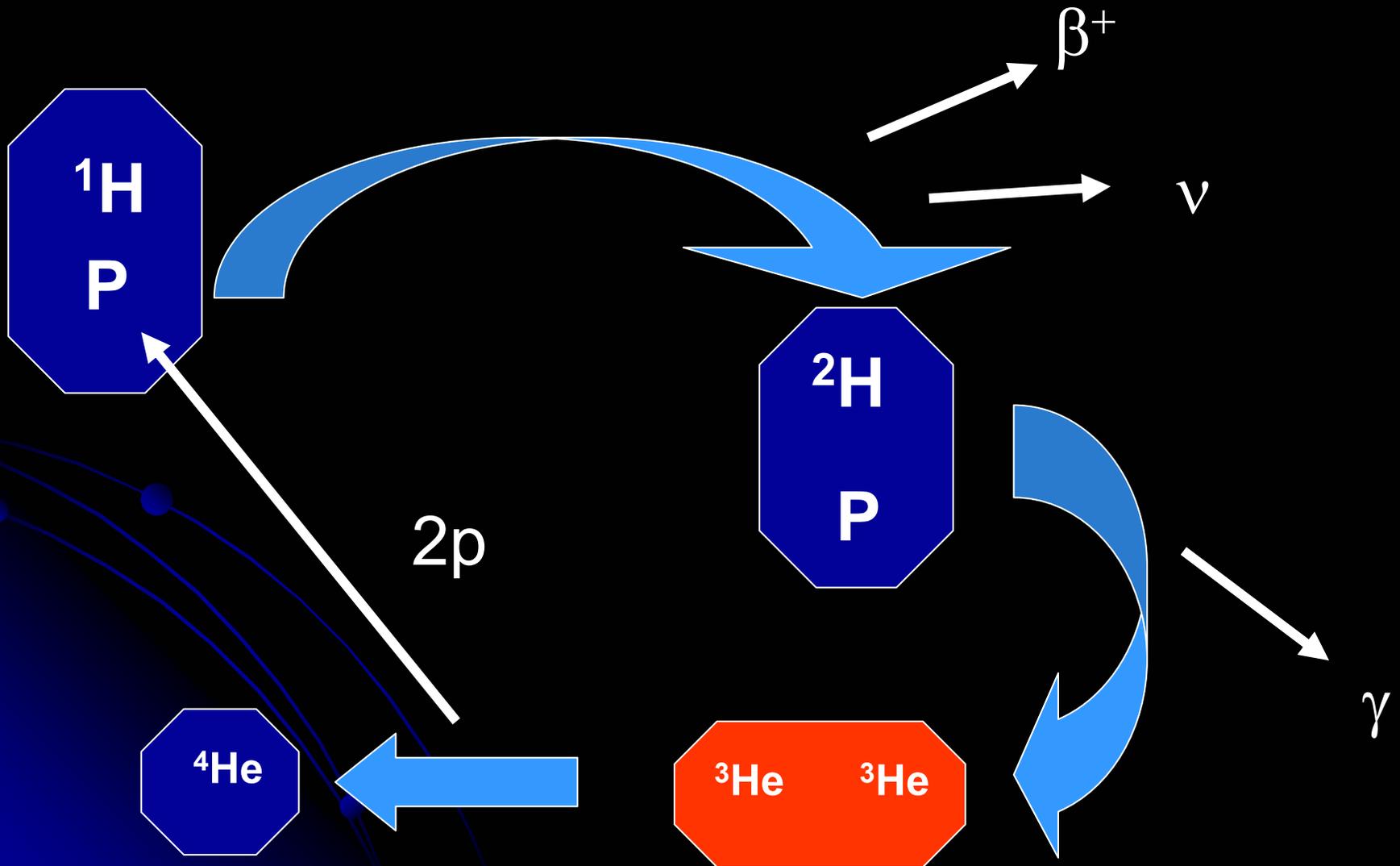
- Real stars are composed of several polytropic layers. For example red giant has isothermal ($n=4$) helium core surrounded by convective ($n=1.5$) hydrogen envelope. One can model such a star by appropriate matching these two phases at some radius.
- 

Nuclear reactions in stars

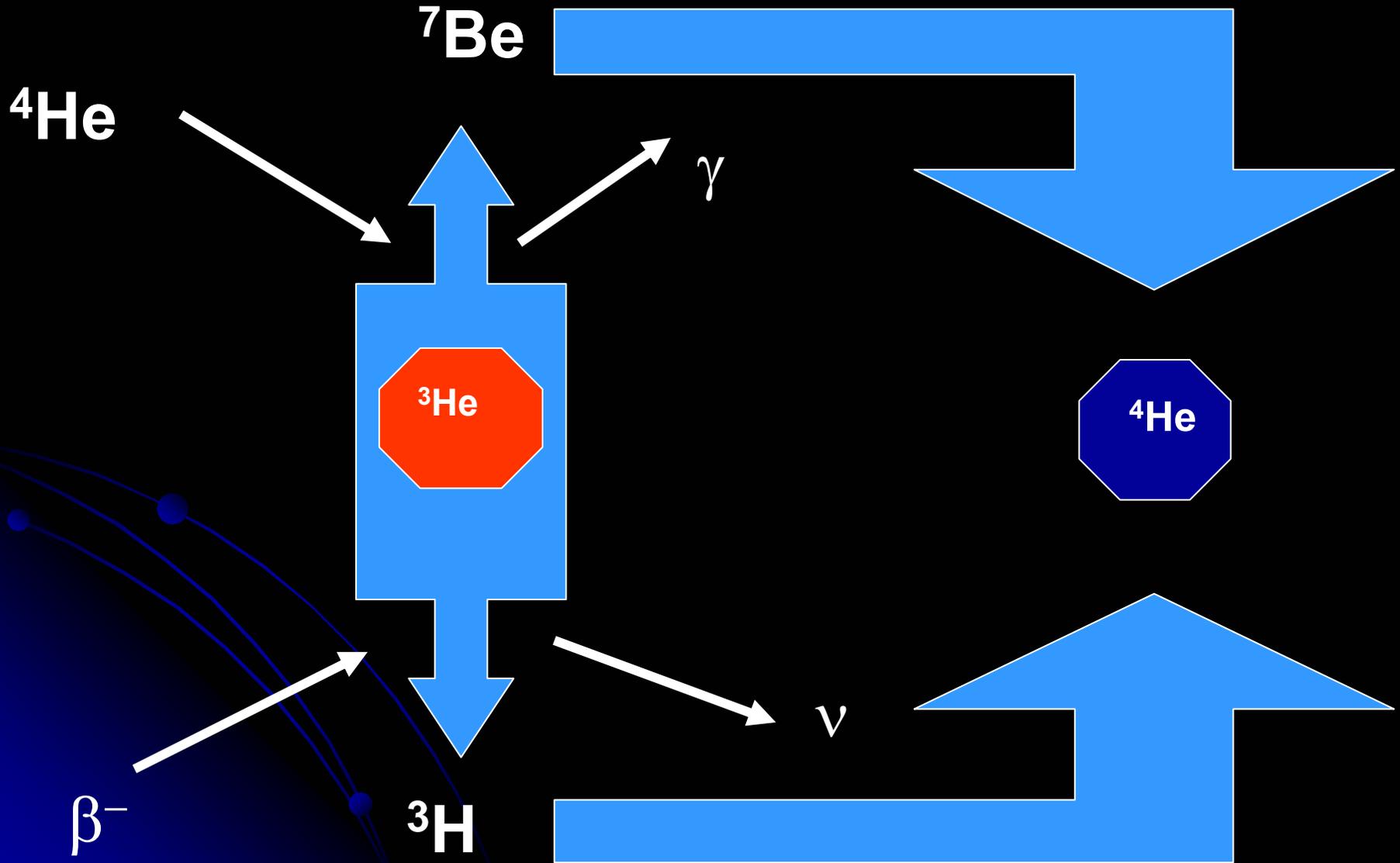
- There are two major processes that are sources of energy in stars: p-p cycle and CNO cycle.
- The effectiveness of a process can be measured by amount of energy produced by one gram of stellar material in unit time

$$\varepsilon = \varepsilon_0 \rho T^\nu$$

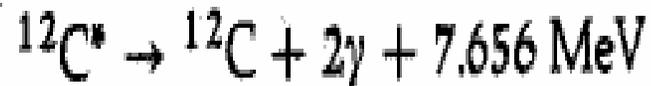
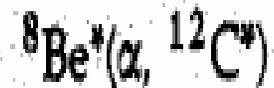
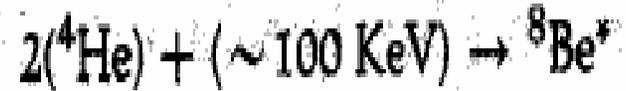
p-p cycle



Other p-p cycles



Triple α process



1.94	$^{12}\overset{\gamma}{\text{C}}(p, \gamma)^{13}\text{N}$	1
1.51	$^{13}\text{N} \rightarrow ^{13}\text{C} + \beta^+ + \nu$	2*
7.54	$^{13}\text{C}(p, \gamma)^{14}\text{N}$	3
7.29	$^{14}\text{N}(p, \gamma)^{15}\text{O}$	4
1.76	$^{15}\text{O} \rightarrow ^{15}\text{N} + \beta^+ + \nu$	5*
4.96	$\left[\begin{array}{l} ^{15}\text{N}(p, \alpha)^{12}\text{C} \\ ^{15}\text{N}(p, \gamma)^{16}\text{O} \end{array} \right]$	6
	$^{16}\overset{\gamma}{\text{O}}(p, \gamma)^{17}\text{F}$	7
	$^{17}\text{F} \rightarrow ^{17}\text{O} + \beta^+ + \nu$	8*
	$\left[\begin{array}{l} ^{17}\text{O}(p, \alpha)^{14}\text{N} \\ ^{17}\text{O}(p, \gamma)^{18}\text{F} \end{array} \right]$	9
	$^{18}\text{F} \rightarrow ^{18}\text{O} + \beta^+ + \nu$	10*
	$\left[\begin{array}{l} ^{18}\text{O}(p, \alpha)^{15}\text{N} \\ ^{18}\text{O}(p, \gamma)^{19}\text{F} \end{array} \right]$	11
	$^{19}\text{F}(p, \alpha)^{16}\text{O}$	12

$$\varepsilon = \varepsilon_0 \rho T^\nu$$

Table 3.4 Temperature Dependence of ν and ε_0

Proton-Proton			CNO Cycle		Triple- α Process		
T_6	ε_0 (cgs)	ν	ε_0 (cgs)	ν	T_8	ε_0 (cgs)	ν
10	7×10^{-2}	4.60	3×10^{-4}	22.9	0.8	2×10^{-12}	49
20	1	3.54	4.5×10^2	18	1.0	4×10^{-8}	41
40	9	2.72	3×10^7	14.1	2.0	15	19
80	43	2.08	2×10^{11}	11.1	3.0	6×10^3	12
100	—	—	2×10^{12}	10.2	4.0	10^5	7.9

- It is a general property of these types of reaction rates that the temperature dependence "weakens" as the temperature increases. At the same time the efficiency ϵ_0 increases. In general, the efficiency of the nuclear cycles rate is governed by the *slowest* process taking place. In the case of p-p cycles, this is always the production of deuterium given in step 1. For the CNO cycle, the limiting reaction rate depends on the temperature. At moderate temperatures, the production of ^{15}O (step 4) limits the rate at which the cycle can proceed. However, as the temperature increases, the reaction rates of all the capture processes increase, but the steps involving inverse β decay (particularly step 5), which do not depend on the state variables, do not and therefore limit the reaction rate. So there is an upper limit to the rate at which the CNO cycle can produce energy independent of the conditions which prevail in the star.

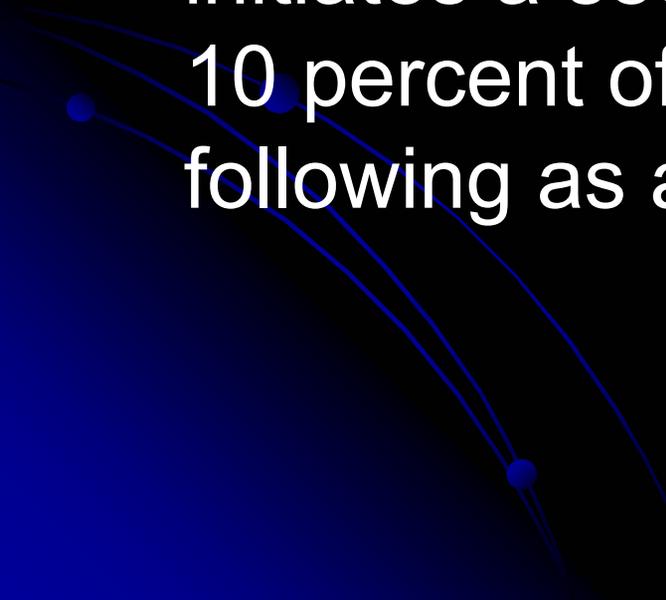
Collapse of protostar

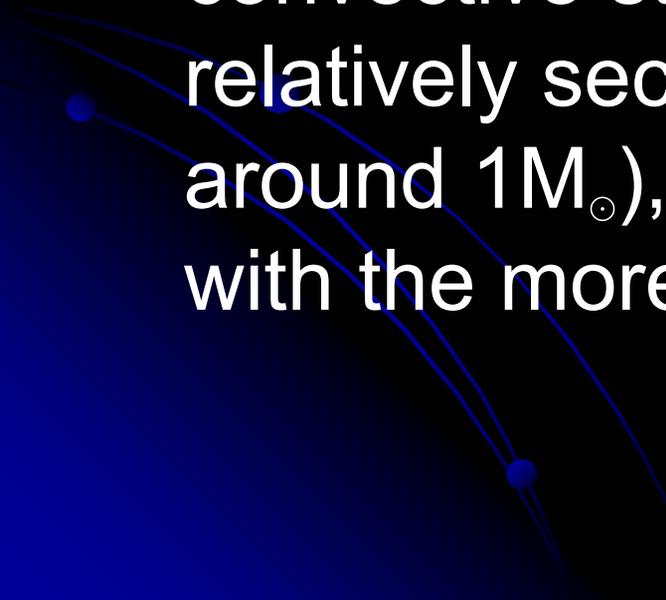
- If the cloud is gravitationally confined within a sphere of the Jeans' length, the cloud will experience rapid core collapse until it becomes optically thick. If the outer regions contain dust, they will absorb the radiation produced by the core contraction and reradiate it in the infrared part of the spectrum. After the initial free-fall collapse of a $1M_{\odot}$ cloud, the inner core will be about 5 AU surrounded by an outer envelope about 20000 AU.

Jeans length

$$2 \left(\frac{3\rho kT}{2\mu m_h} \right) \left(\frac{4\pi R_c^2}{3} \right) \leq \frac{GM^2}{R_c}$$


$$R_J = \frac{GM \mu m_h}{3kT}$$

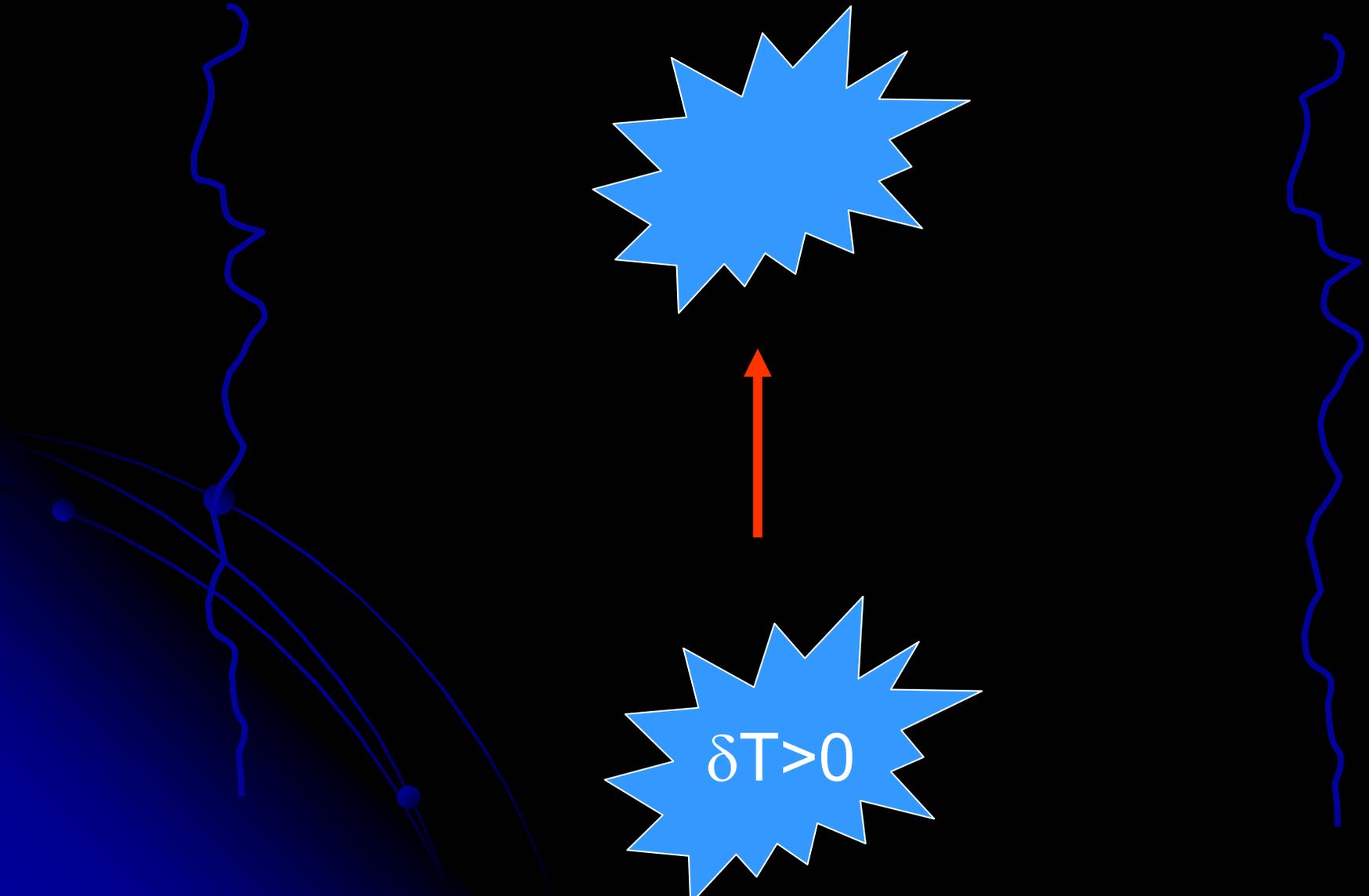
- When the core temperature reaches about 2000 K, the ^2H molecules dissociate, thereby absorbing a significant amount of the internal energy. The loss of this energy initiates a second core collapse of about 10 percent of the mass with the remainder following as a "heavy rain".
- 

- After a time, sufficient matter has rained out of the cloud, and the cloud becomes relatively transparent to radiation and falls freely to the surface, producing a fully convective star. While this scenario seems relatively secure for low mass stars (i.e., around $1M_{\odot}$), difficulties are encountered with the more massive stars.
- 

Contraction onto Main Sequence

- Once the protostar has become opaque to radiation, the energy liberated by the gravitational collapse of the cloud cannot escape to interstellar space. The collapse slows down dramatically and the future contraction is limited by the star's ability to transport and radiate the energy away into space. Hayashi showed that there would be a period after the central regions became opaque to radiation during which the star would be in convective equilibrium.

Convection by buoyancy



Hayashi Evolutionary Tracks

- Once convection is established, it is incredibly efficient at transporting energy. Thus, as long as there are no sources of energy other than gravitation, the future contraction is limited by the star's ability to radiate energy into space rather than by its ability to transport energy to the surface. The structure of a fully convective star is that of a polytrope of index $n = 1.5$. We may combine these two properties of the star to approximately trace the path it must take on the Hertzsprung-Russell diagram.

- Some of the energy generated by contraction will be released from the stellar surface in the form of photons. As long as the process is slow, the virial theorem will hold and $\langle T \rangle \approx 0$. Thus

$$\frac{1}{2} \langle \Omega \rangle = - \langle U \rangle$$

- This implies that one-half of the change in the gravitational energy will go into raising the internal kinetic energy of the gas. The other half is available to be radiated away.

- Therefore

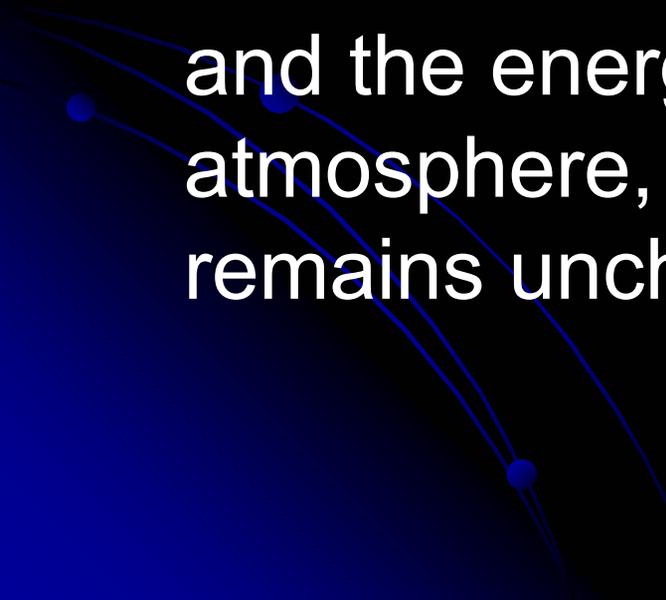
$$L = \frac{1}{2} \frac{d}{dt} \left(\frac{GM^2}{R} \right) = -\frac{1}{2} \frac{GM^2}{R^2} \frac{dR}{dt}$$

- Since the luminosity is related to the surface parameters by (def. of effective temperature)

$$L = 4\pi R_*^2 \sigma T_e^4$$

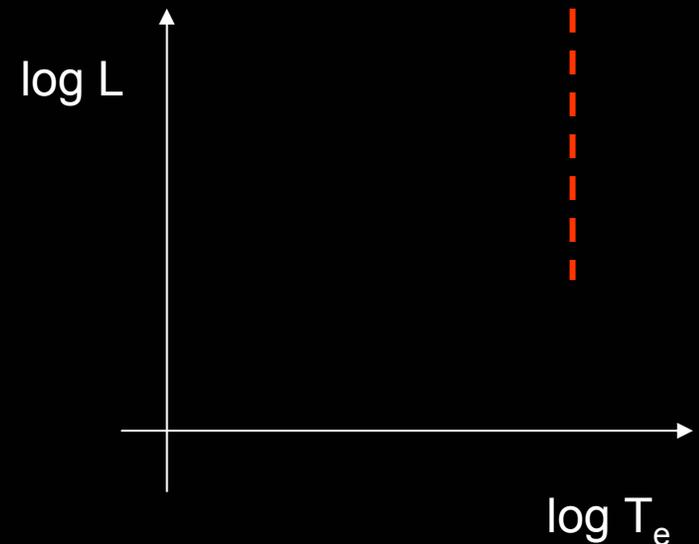
- the change in the luminosity with respect to the radius will be

$$\frac{dL}{dR_*} = \frac{4L}{T_e} \frac{dT_e}{dR_*} + \frac{2L}{R_*}$$

- As long as there are no sources of energy other than gravitation, the contraction is limited by the star's ability to radiate energy into space rather than by its ability to transport energy to the surface.
 - So as long as the stellar luminosity is determined solely by the change in gravity, and the energy loss is dictated by the atmosphere, we might expect that T_e remains unchanged.
- 

- Thus dT_e/dR_* is approximately zero, and we expect the star to move vertically down the Hertzsprung-Russell (H-R) diagram with the luminosity changing roughly as R_*^2 until the internal conditions within the star change. For the Hayashi tracks

$$\frac{dT_e}{dR_*} = \frac{dT_e}{dL} = 0, \quad \frac{d \ln L}{d \ln R_*} = 2$$



- While the location of a specific track depends on the atomic physics of the photosphere, the relative location of these tracks for stars of differing mass is determined by the fact that the star is a polytrope of index $n = 3/2$. From the polytropic mass-radius relation


$$M^{1/3} R_* = \text{const} \quad \Rightarrow \quad \frac{d \log R_*}{d \log M} = -\frac{1}{3}$$

$$L = 4\pi R_*^2 \sigma T_e^4$$

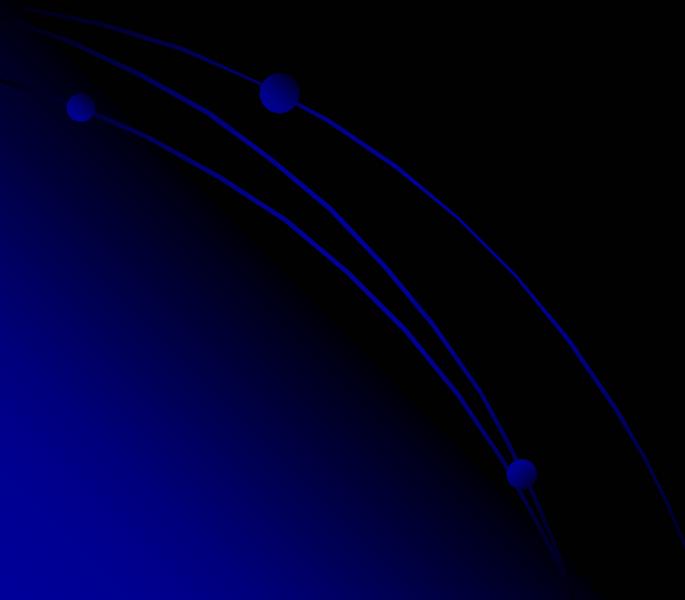
$$\frac{dL}{dR_*} = \frac{4L}{T_e} \frac{dT_e}{dR_*} + \frac{2L}{R_*} \quad \Rightarrow \quad \frac{dL}{dM} = \frac{2L}{R_*} \frac{dR_*}{dM} + \frac{4L}{T_e} \frac{dT_e}{dM}$$

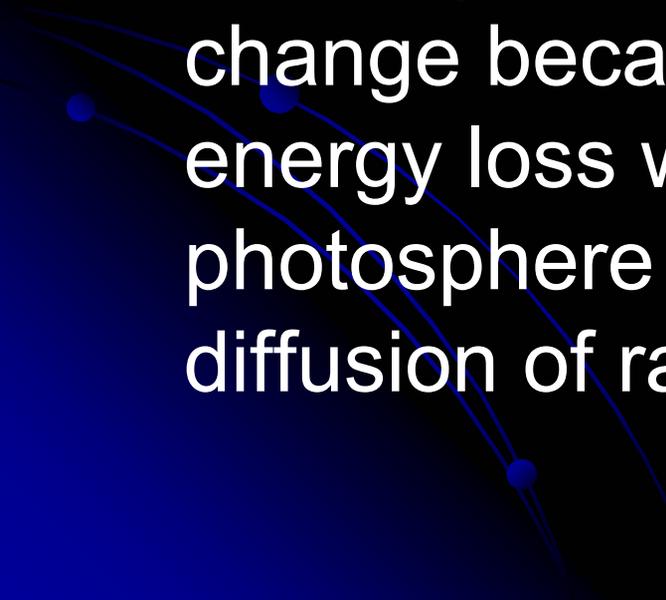
- If we inquire as to the spacing of the vertical Hayashi tracks in the H-R diagram, then we can look for the effective temperatures for stars of different mass but at the same luminosity.

$$\frac{d \log T_e}{d \log M} = \frac{1}{6}$$

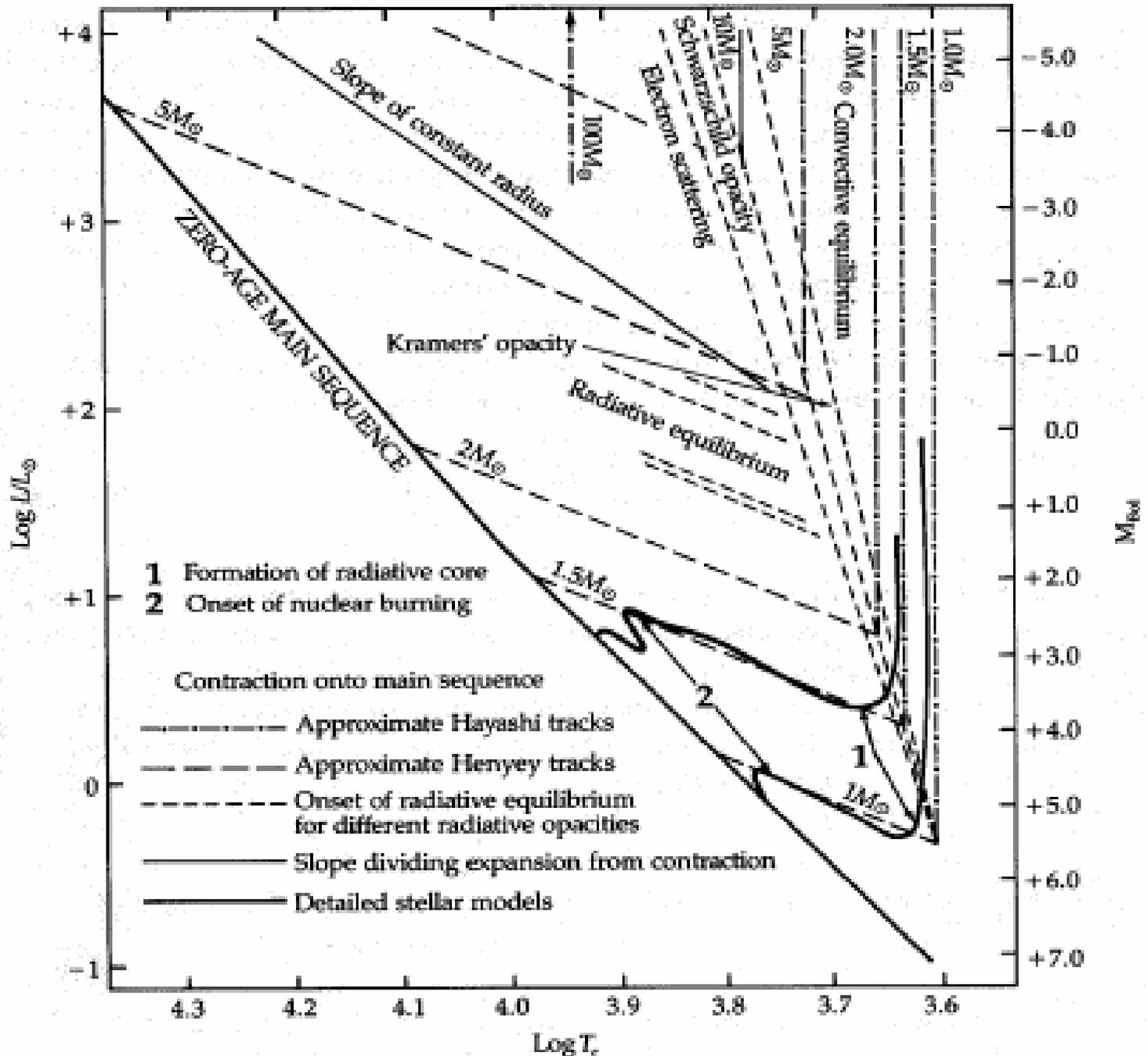
- This extremely weak dependence of the effective temperature on mass means that we should expect all the Hayashi tracks for the majority of main sequence stars to be bunched on the right side of the H-R diagram. Since the star is assumed to be radiating as a blackbody of a given T_e and is in convective equilibrium, no other stellar configuration could lose its energy more efficiently. Thus no stars should lie to the right of the Hayashi track of the appropriate mass on the H-R diagram; this is known as the *Hayashi zone of avoidance*.

- We may use arguments like these to describe the path of the star on the H-R diagram followed by a gravitationally contracting fully convective star. This contraction will continue until conditions in the interior change as a result of continued contraction.



- As the star moves down the Hayashi track, the internal temperature increases so that $T = \mu M/R$.
 - At some point, depending on the dominant source of opacity, and convection will cease.
 - At that point the mode of collapse will change because the primary barrier to energy loss will move from the photosphere to the interior and the diffusion of radiant energy.
- 

- As the star continues to shine, the gravitational energy continues to become more negative, and to balance it, in accord with the virial theorem, the internal energy continues to rise. This results in a slow but steady increase in the temperature gradient which results in a steady increase in the luminosity as the radiative flux increases. This increased luminosity combined with the ever-declining radius produces a sharply rising surface temperature as the photosphere attempts to accommodate the increased luminosity. This will yield tracks on the H-R diagram which move sharply to the left while rising slightly.



- We may quantify this by asking how the luminosity changes in time.

$$\frac{dL}{dt} = -\frac{1}{2} \frac{d^2\Omega}{dt^2} = -\frac{1}{2} \frac{GM^2}{R_*^2} \left(-\frac{2}{R_*} \left(\frac{dR_*}{dt} \right)^2 + \frac{d^2R_*}{dt^2} \right)$$

- If we further invoke the virial theorem and require that the contraction proceed so as to keep the second derivative of the moment of inertia equal to zero, then

$$\frac{d^2I}{dt^2} = \frac{d^2}{dt^2} (\alpha MR_*^2) = 0 \quad \Rightarrow \quad \left(\frac{dR_*}{dt} \right)^2 + R_* \frac{d^2R_*}{dt^2} = 0$$

Thus (Henyeey evolution)

$$\frac{d \log L}{d \log R_*} = -3$$

- Moreover

$$\frac{dL}{dM} = \frac{2L}{R_*} \frac{dR_*}{dM} + \frac{4L}{T_e} \frac{dT_e}{dM} \quad \parallel \quad \left(\begin{array}{c} R_* \frac{dM}{dR_*} \\ M \frac{dT_e}{dM} \end{array} \right) \Rightarrow$$

$$\frac{d \log T_e}{d \log R_*} = -\frac{5}{4}, \quad \frac{d \log L}{d \log T_e} = \frac{12}{5}$$

