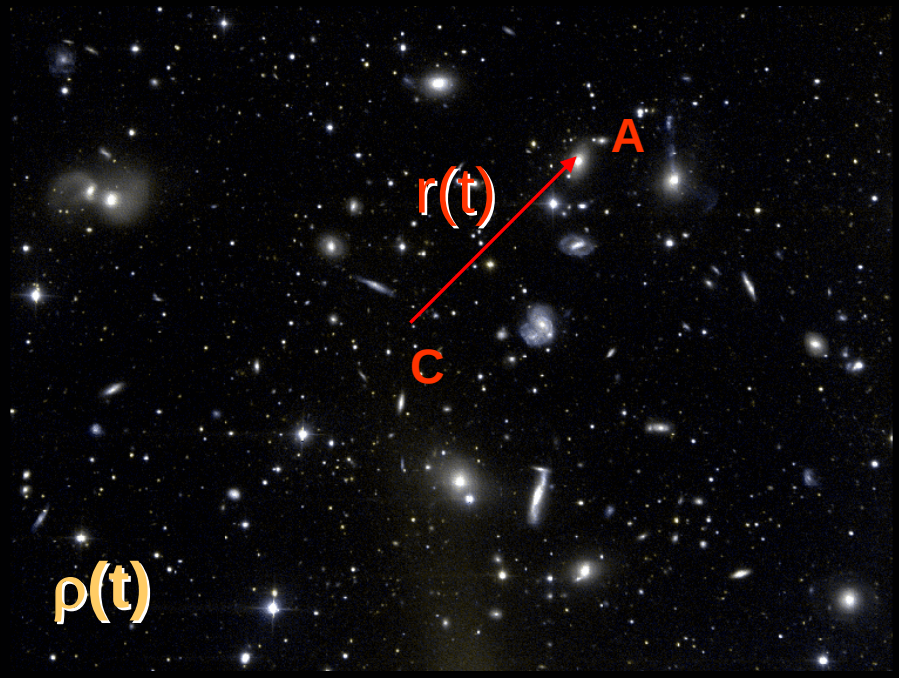
Relativistic Universe

Relativistic cosmology

Newtonian cosmology

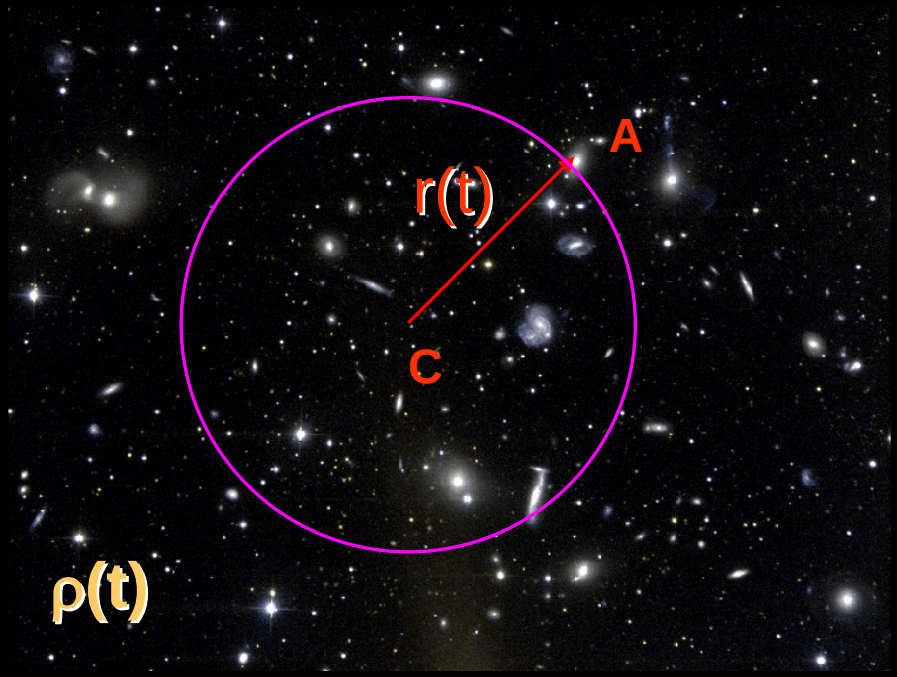
- Consider a universe formed by a very large cloud of galaxies.
- Assume, in accordance with cosmological principle, that galaxies are – and remain – uniformly distributed in space with time dependent density $\rho(t)$, and that they attract each other gravitationally.
- Let C be the center of the cloud. Consider a galaxy A at the distance r(t) from the center C.



- The gravitational force on A due to the galaxies outside the sphere cancels out, and the gravitational force due to the galaxies inside the sphere is the same as the force due to the same mass concentrated at C. Thus

$$F = -G\frac{m_A \frac{4}{3}\pi r^3(t)\rho(t)}{r^2(t)};$$

$$\frac{d^2 r}{dt^2} = -G\frac{4}{3}\pi r(t)\rho(t)$$



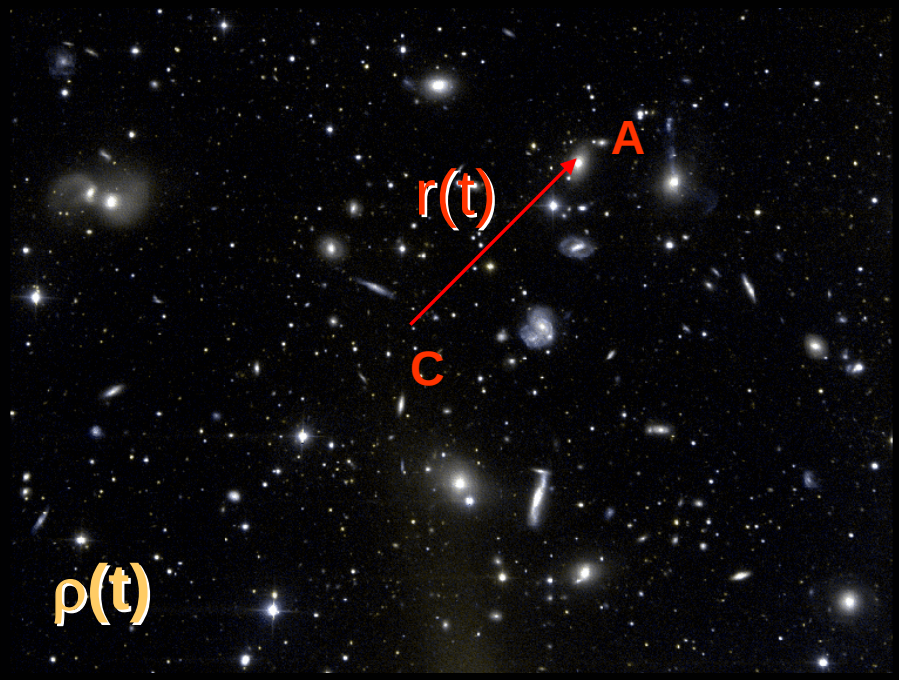
- If the density remains spatially constant, $\rho(t)=\rho_0 r^{-3}(t)$, where ρ_0 is the density at $r(t)=1$.

$$\frac{d^2 r}{dt^2} = -\frac{4}{3}\pi G \rho_0 \frac{1}{r^2(t)} = -\frac{c}{r^2(t)}$$

- This is the cosmological equation in GR as well, for nonrelativistic matter and spatially flat universe.

Could you tell, by local experiments, if you are in the inertial galaxy C, or in the accelerated one A?

- Suppose I perform an experiment at A. Three forces act on my test body: local force, F_{local}, inertial force, $F_{inertial}$, and the cosmological pull towards C, $F_{cosmological}$. But the last two forces cancel exactly!
- The only thing I observe is that the universe expands.

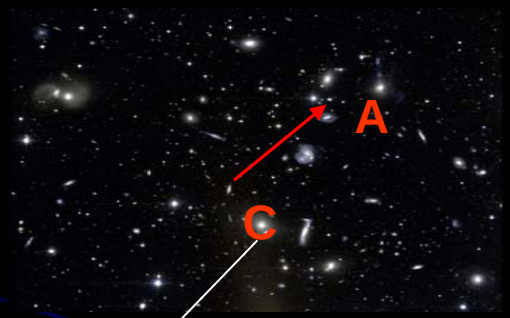


Friedman-Robertson-Walker metric

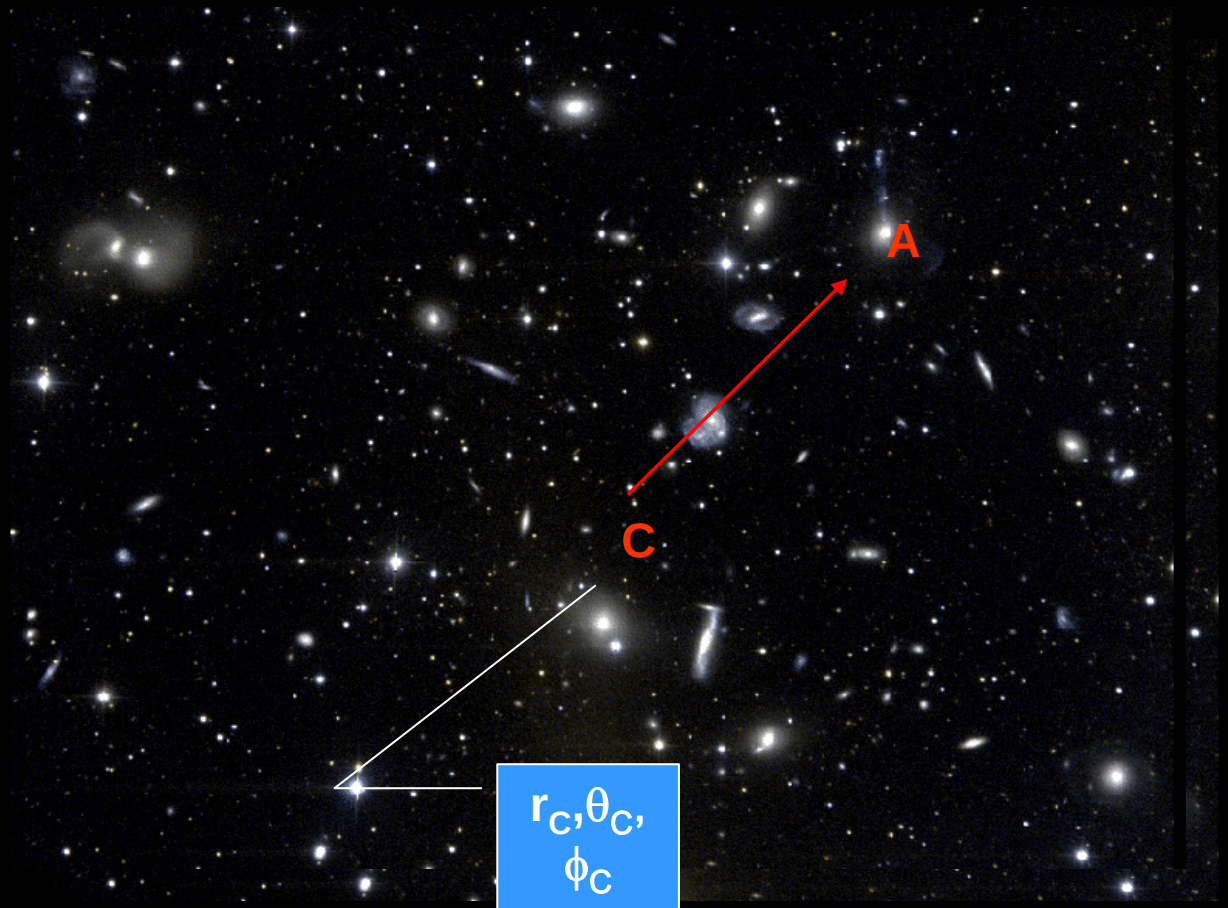
$$ds^2 = (c\,dt)^2 - a(t)^2 \left[\frac{dr^2}{1 - Kr^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right]$$

- r , θ and ϕ are the comoving coordinates (r is by convention dimensionless);
- t is the proper time;
- $a(t)$ is a function to be determined which has the dimensions of a length and is called the *cosmic scale factor* or the *expansion parameter*;
- the *curvature parameter* K is a constant which can be scaled in such a way that it takes only the values 1, 0 or −1.

The meaning of comoving coordinates: the same for all points at all times



r_C, θ_C, ϕ_C



r_C, θ_C, ϕ_C

FRW metric

$$ds^2 = (c\,dt)^2 - a(t)^2 \left[\frac{dr^2}{1-Kr^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right]$$

- Let us try to understand the spatial part of the FRW metric better. Let us fix time to be constant t=const. Then we have the metric on constant time slice.

$$ds^2_{t=const} = -a^2 \left[\frac{dr^2}{1-Kr^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right]$$

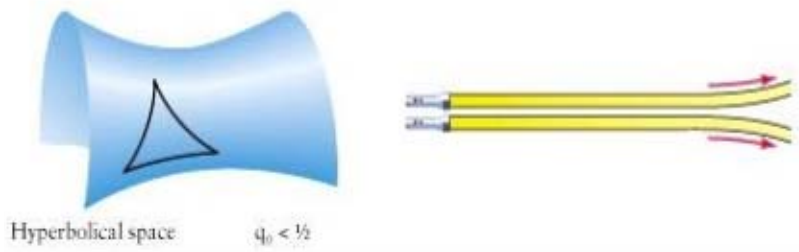
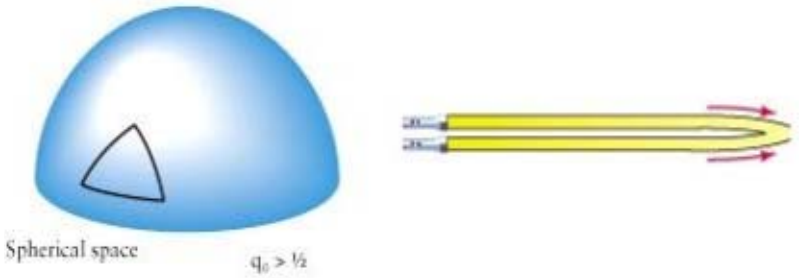
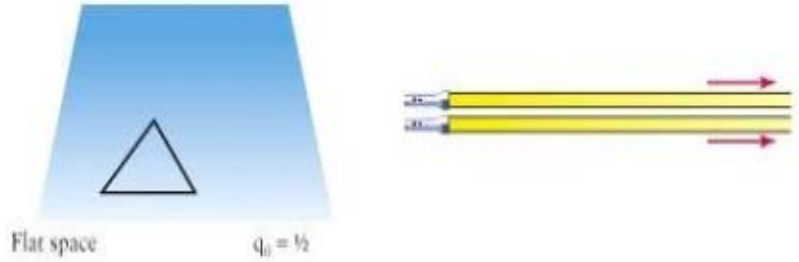
2D sphere of radius *a*

$$dl^2 = a^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Introducing (dimensionless) coordinate r = sin θ

$$dl^2 = a^2\left(\frac{dr^2}{1-r^2} + r^2\, d\phi^2\right)$$

$$dl^2 = a^2\left(\frac{dr^2}{1-Kr^2} + r^2\,d\phi^2\right)$$

Type	Shape of Universe
Open	
Closed	
Flat	

$$dl^2 = a^2\left(\frac{dr^2}{1+r^2} + r^2\,d\phi^2\right)$$

$$dl^2 = a^2\left(\frac{dr^2}{1-r^2} + r^2\,d\phi^2\right)$$

$$dl^2 = a^2\left(dr^2 + r^2\,d\phi^2\right)$$

FRW cosmology

- In homogeneous, isotropic universe, matter content is characterized by density $\rho(t)$ and pressure $p(t)$. According to the cosmological principle both depend on time only!

Types of matter

- To solve Einstein cosmological equations, we must know what type of matter fills the universe in:

1. Dust (nonrelativistic, cold): $p=0;$
2. Radiation (relativistic, hot): $p=1/3\ \rho\ ;$
3. Dark energy (e.g., cosmological constant): $p= -\ \rho\ ;$

- (Mixtures are possible)

FRW cosmology - equations

$$\ddot{a}(t) = -\frac{4}{3}\pi G\left(\rho(t) + 3\frac{p(t)}{c^2}\right)a(t)$$

$$\dot{a}^2(t) + Kc^2 = \frac{8}{3}\pi G\rho(t)a^2(t)$$

Combining these equations we get energy conservation

$$\dot{\rho}(t) + 3\left(\rho(t) + \frac{p(t)}{c^2}\right)\frac{\dot{a}(t)}{a(t)} = 0$$

Curvature parameter

$$\frac{K}{a^2(t)} = \frac{1}{c^2}\frac{\dot{a}^2(t)}{a^2(t)}\left(\frac{\rho(t)}{\rho_c(t)} - 1\right); \quad \rho_c(t) \equiv \frac{3}{8\pi G}\frac{\dot{a}^2(t)}{a^2(t)}$$

- ρ_c is called the *critical density*. The space is closed (K = 1), flat (K = 0) or open (K = −1) according to whether the *density parameter* $\Omega = \rho/\rho_c$ is greater, equal, or less than 1.

The Hubble law

- The *proper distance*, d_P, of a point P from another point P_0, which we take to define the origin of a set of polar coordinates r , ϑ and φ, is the distance measured by a chain of rulers held by observers which connect P to P_0 at time t. From the FRW with dt = 0 this can be seen to be

$$d_P = \int_0^r \frac{a\,dr'}{\sqrt{1-Kr'^2}} = a\,f(r)$$

$$f(r) = \sin^{-1} r \qquad (K=1);$$
$$f(r) = r \qquad (K=0);$$
$$f(r) = \sinh^{-1} r \qquad (K=-1)$$

- Of course this proper distance is of little operational significance because one can never measure simultaneously all the distance elements separating P from P_0. The proper distance at time t is related to that at the present time t_0 by

$$d_P(t_0) = a_0 f(r) = \frac{a_0}{a} d_P(t)$$

- a_0 is the value of $a(t)$ at $t = t_0$. Instead of the comoving coordinate r one could also define a radial comoving coordinate of P by the quantity

$$d_c = a_0 f(r) \quad \Rightarrow \quad d_c = \frac{a_0}{a} d_P$$

The Hubble law

- The proper distance d_P of a source may change with time because of the time dependence of the expansion parameter a. In this case a source at P has a radial velocity with respect to the origin P_0 given by

$$v_r = \dot{a}\, f(r) = \frac{\dot{a}}{a} d_P = H d_P;$$

$$H = \frac{\dot{a}}{a} \cong 72\ km\ s^{-1}\ Mpc^{-1}$$

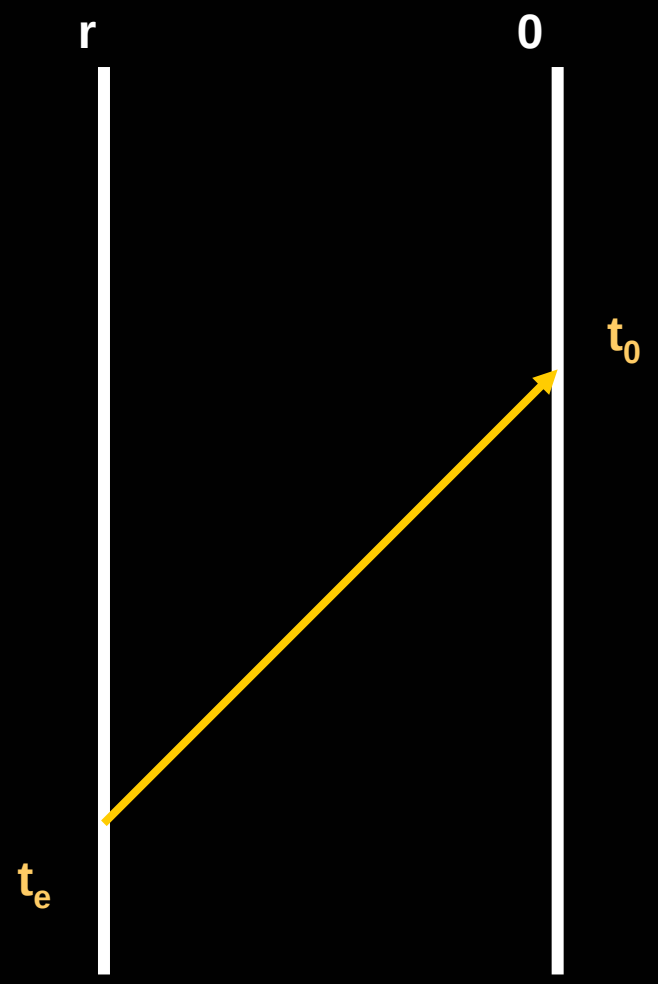
The redshift

- It is useful to introduce a new variable related to the expansion parameter *a(t)* which is more directly observable. We call this variable the *redshift z.* Many of the relevant formulae are very simple when expressed in terms of this variable.

$$z = \frac{\lambda_0 - \lambda_e}{\lambda_e}$$

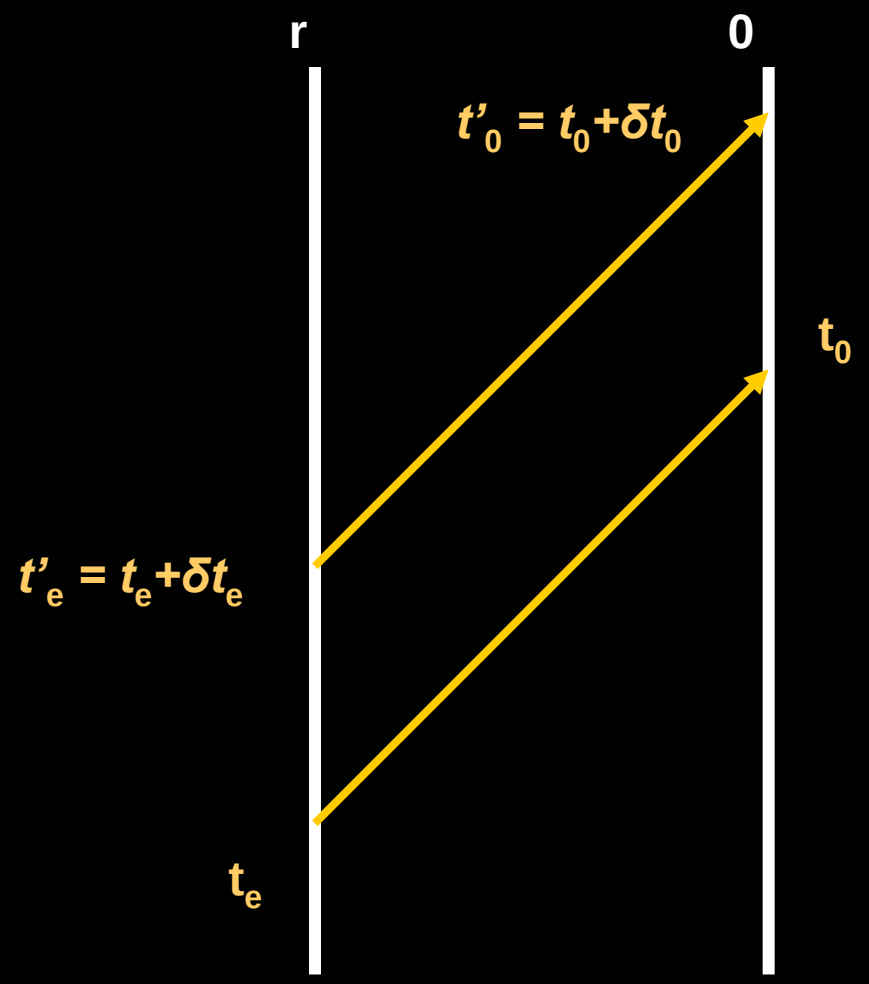
- λ_0 is the wavelength of radiation from the source observed at O (which we take to be the origin of our coordinate system) at time t_0 and emitted by the source at some (earlier) time t_e; the source is moving with the expansion of the universe and is at a comoving coordinate r . The wavelength of radiation emitted by the source is λ_e.

- The radiation travels along a light ray (null geodesic) from the source to the observer so that $ds^2 = 0$ and, therefore,



$$\int_{t_e}^{t_0} \frac{c\,dt}{a(t)} = \int_0^r \frac{dr}{\sqrt{1 - Kr^2}} = f(r)$$

- Light emitted from the source at $t'_e = t_e + \delta t_e$ reaches the observer at $t'_0 = t_0 + \delta t_0$.
- Given that *f(r)* does not change, because *r* is a comoving coordinate and both the source and the observer are moving with the cosmological expansion, we have



$$\int_{t'_e}^{t'_0} \frac{c\,dt}{a(t)} = f(r)$$

$$\frac{\delta t_0}{a_0} = \frac{\delta t_e}{a_e}$$

$$\delta t_0 = \frac{1}{\nu_0} \sim \lambda_0; \;\; \delta t_e = \frac{1}{\nu_e} \sim \lambda_e$$

$$\frac{\lambda_0}{a_0} = \frac{\lambda_e}{a_e}$$

$$z + 1 = \frac{a_0}{a_e}$$

Deceleration parameter

- The Hubble parameter *H(t)* measures the expansion rate at any particular time *t* for any model obeying the Cosmological Principle. It does, however, vary with time in a way that depends upon the contents of the Universe. One can express this by expanding the cosmic scale factor for times *t* close to t_0 in a power series

$$a(t) = a_0 \left[1 + H_0 (t - t_0) - \frac{1}{2} q_0 H^2{}_0 (t - t_0)^2 + \ldots \right]$$

$$q_0 = -\frac{\ddot{a}(t_0) a(t_0)}{\dot{a}(t_0)^2}$$

Redshift

$$z+1=\frac{a_0}{a_e}$$

$$z=H_0(t-t_0)+\left(1+\frac{1}{2}q_0\right)H^2{}_0(t-t_0)^2+\ldots$$

$$(t-t_0)=\frac{1}{H_0}\left[z-\left(1+\frac{1}{2}q_0\right)z^2+\ldots\right]$$

r as a function of *z*

$$\int_{t_e}^{t_0} \frac{c\,dt}{a(t)} = \int_0^r \frac{dr}{\sqrt{1-Kr^2}}$$

$$\frac{c}{a_0}\int_t^{t_0}\left[1+H_0(t-t_0)+\left(1+\frac{1}{2}q_0\right)H^2{}_0(t-t_0)^2\right]dt = r+O(r^3)$$

$$r=\frac{c}{a_0H_0}\left[z-\left(1+\frac{1}{2}q_0\right)z^2+\ldots\right]$$

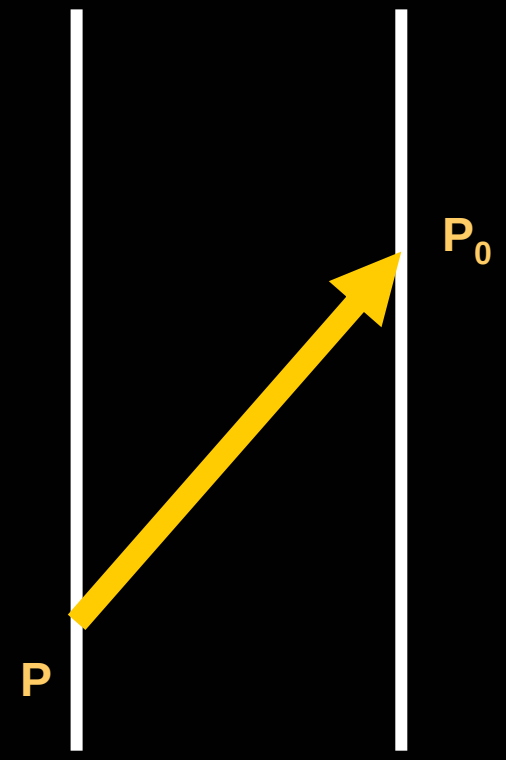
Expressions of this type are useful because they do not require full solutions of the Einstein equations for *a(t)*; the quantity q_0 is used to parametrise a family of approximate solutions for *t* close to t_0.

Cosmological distances

- The comoving coordinate system relates to *proper distance* (i.e. distances measured in a hypersurface of constant proper time) in spaces described by the FRW metric. However, we cannot measure proper distances to astronomical objects in any direct way. Distant objects are observed only through the light they emit which takes a finite time to travel to us; we cannot therefore make measurements along a surface of constant proper time, but only along the set of light paths travelling to us from the past – our past *light cone*. One can, however, define operationally other kinds of distance which are directly measurable.

Luminocity distance

- One such distance is the *luminosity distance* d_L. This is defined in such a way as to preserve the Euclidean inverse-square law for the diminution of light with distance from a point source. Let L denote the power emitted by a source at a point P, which is at a coordinate distance r at time t. Let l be the power received per unit area (i.e. the flux) at time t_0 by an observer placed at P_0. We then define



$$d_L = \sqrt{\frac{L}{4\pi l}}$$

- The area of a spherical surface centred on P and passing through P_0 at time t_0 is just $4\pi a^2_0 r^2$. The photons emitted by the source arrive at this surface having been redshifted by the expansion of the universe by a factor a/a_0. Also photons emitted by the source in a small interval δt arrive at P_0 in an interval $\delta t_0 = (a_0/a)\delta t$ due to a time-dilation effect. Thus

$$l = \frac{L}{4\pi a_0{}^2 r^2}\left(\frac{a}{a_0}\right)^2$$

$$d_L = a_0{}^2 \frac{r}{a}$$

$$d_L = \frac{c}{H_0}\left[z - \frac{1}{2}(1-q_0)z^2 + \ldots\right]$$

The m-z and N-z relations

- The general relationship between redshift and distance could be used to probe its geometry and to test the Cosmological Principle.
- If galaxy clusters, quasars, etc. are distributed homogeneously and isotropically, it is interesting to consider two relationships:
- *m–z* relationship between the apparent magnitude of a source and its redshift and
- *N(> l)–z* relationship between the number of sources of a given type with apparent luminosity greater than some limit *l* and redshift less than *z*.
- These relations may also provide a way of determining the deceleration parameter q_0.

m-z relation

- Astronomers work with the following quantities : the *absolute magnitude M* and the *apparent magnitude m.* The magnitude scale is defined logarithmically by taking a factor of **100** in received flux to be a difference of **5** magnitudes. The zero-point can be fixed in various ways. The absolute magnitude is defined to be the apparent magnitude the source would have if it were placed at a distance of 10 parsec. The relationship between the luminosity distance of a source, its apparent magnitude *m* and its absolute magnitude *M* is

$$d_L = 10^{1+(m-M)/5}\,\mathrm{pc}$$

- The quantity is called *distance modulus*

$$m - M = -5 + 5\log d_L(\mathrm{pc})$$

- In the regime where it is accurate, for $z < 0.2$, can provide an estimate of H_0, together with a strong confirmation of the validity of the Hubble law and of the Cosmological Principle.

$$d_L = \frac{c}{H_0}\left[z - \frac{1}{2}(1 - q_0)z^2 + \ldots\right]$$

$$m - M \cong 25 - 5\log H_0 + 5\log cz + 1.086(1 - q_0 z)$$