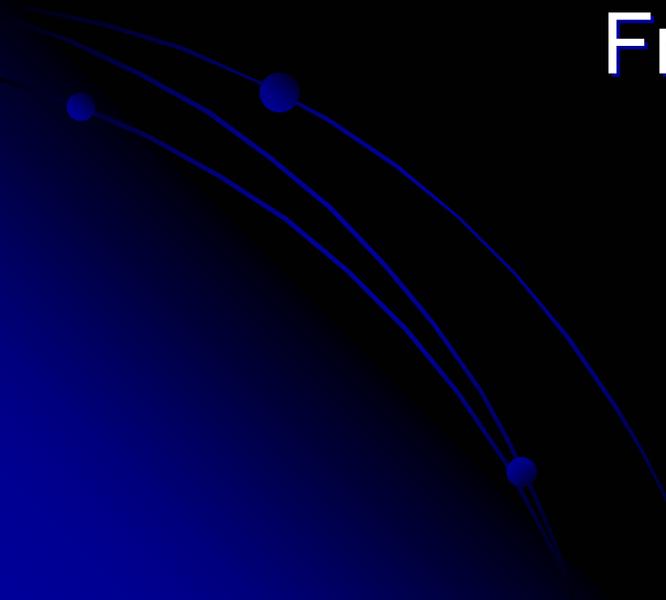


# Relativistic Universe

Friedmann models



# Perfect fluid models

- The ideal perfect fluid is a realistic approximation in many situations. For example, if the mean free path between particle collisions is much less than the scales of physical interest, then the fluid may be treated as perfect.
- We need to specify an equation of state for our fluid in the form  $p = p(\rho)$ .

$$p = w\rho c^2, \quad 0 \leq w \leq 1 \quad (\text{Zeldovich interval})$$

# Dust

- The case with  $w = 0$  represents *dust* (pressure less material). This is also a good approximation to the behavior of any form of non-relativistic fluid or gas.
- For ideal gas  $w = 0$  for temperatures not too high

$$p = nkT = \frac{kT}{mc^2} \rho c^2 = w(T) \rho c^2$$

# Radiation

- At the other extreme, a fluid of non-degenerate, ultrarelativistic particles in thermal equilibrium has an equation of state with  $w = 1/3$

$$p = \frac{1}{3} \rho c^2$$

- For instance, this is the case for a gas of photons. A fluid with an equation of state of this type is usually called a *radiative fluid*, though it may comprise relativistic particles of any form.

# Friedmann equations

$$\ddot{a}(t) = -\frac{4}{3}\pi G \left( \rho(t) + 3\frac{p(t)}{c^2} \right) a(t) + \frac{\Lambda c^2 a}{3}$$

$$\dot{a}^2(t) + Kc^2 = \frac{8}{3}\pi G \rho(t) a^2(t) + \frac{\Lambda c^2 a^2}{3}$$

Combining these equations ( $\Lambda = 0$ ) we get energy conservation

$$\dot{\rho}(t) + 3 \left( \rho(t) + \frac{p(t)}{c^2} \right) \frac{\dot{a}(t)}{a(t)} = 0$$

# Friedmann eqs. for fluid

$$\dot{\rho}(t) + 3 \left( \rho(t) + \frac{p(t)}{c^2} \right) \frac{\dot{a}(t)}{a(t)} = 0$$

$$d(\rho a^3) = -3 \frac{p(t)}{c^2} a^2 da$$

$$p(t) = w \rho(t) c^2$$

# Solutions of Friedmann eqs.

$$\rho a^{3(1+w)} = \text{const} = \rho_{0w} a_0^{3(1+w)}$$

- We use the suffix '0' to denote a reference time, usually the present.
- Notice the peculiar case in which  $w = -1$ , which is the perfect fluid equivalent of a cosmological constant. The energy density does not vary as the universe expands for this kind of fluid.
- In particular we have, for a *dust universe* ( $w = 0$ ) or a *matter universe*

$$\rho a^3 = \rho_m a^3 = \text{const} = \rho_{0m} a_0^3$$

- for a *radiative universe* ( $w = 1/3$ )

$$\rho a^4 = \rho_r a^4 = \text{const} = \rho_{0r} a_0^4$$

- If one replaces the expansion parameter  $a$  with the redshift  $z$ , one finds, for dust and non-relativistic matter,

$$\rho_m = \rho_{0m} (1+z)^3$$

- and, for radiation and relativistic matter,

$$\rho_r = \rho_{0r} (1+z)^4$$

- The difference between these equations can be understood quite straightforwardly if one considers a comoving box containing, say,  $N$  particles. Let us assume that, as the box expands, particles are neither created nor destroyed. If the particles are non-relativistic (i.e. if the box contains 'dust'), then the density simply decreases as the cube of the scale factor. On the other hand, if the particles are relativistic, then they behave like photons: not only is their number-density diluted by a factor  $a^3$ , but also the wavelength of each particle is increased by a factor  $a$  resulting in a redshift  $z$ . Since the energy of the particles is inversely proportional to their wavelength the total energy must decrease as the fourth power of the scale factor.

# Big Bang singularity

- Models of the Universe made from fluids with  $-1/3 < w < 1$  have the property that they possess a point in time where  $a$  vanishes and the density diverges. This instant is called the *Big Bang singularity*.

$$\dot{a}^2(t) + Kc^2 = \frac{8}{3} \pi G \rho(t) a^2(t)$$

$$\frac{K}{a^2(t)} = \frac{1}{c^2} \frac{\dot{a}^2(t)}{a^2(t)} \left( \frac{\rho(t)}{\rho_c(t)} - 1 \right); \quad \rho_c(t) \equiv \frac{3}{8\pi G} \frac{\dot{a}^2(t)}{a^2(t)}$$

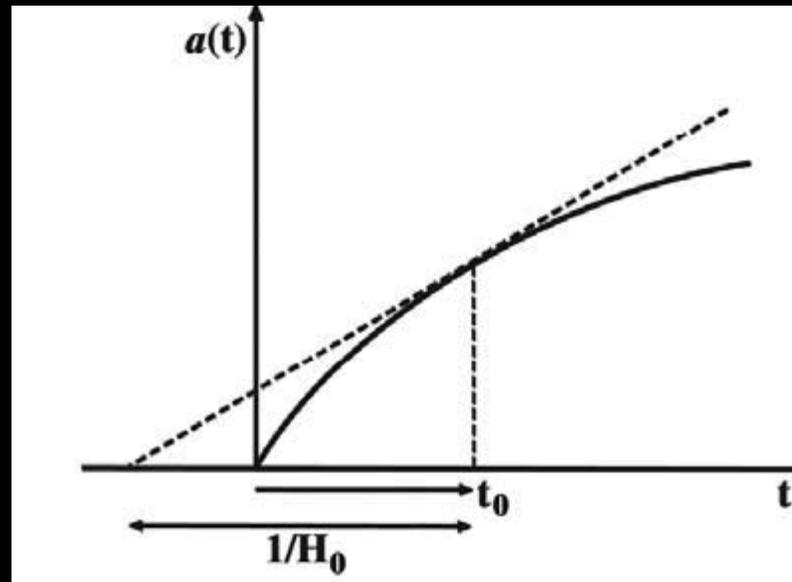
- Suppose at some generic time,  $t$  (for example the present time,  $t_0$ ), the universe is expanding, so that  $\dot{a} > 0$

- Then from

$$\ddot{a}(t) = -\frac{4}{3}\pi G \left( \rho(t) + 3\frac{p(t)}{c^2} \right) a(t)$$

- $\ddot{a} < 0$  for all  $t$  provided

$$\left( \rho(t) + 3\frac{p(t)}{c^2} \right) > 0, \quad (1 + 3w) > 0$$



One can see therefore that  $a(t)$  must be equal to zero at some finite time in the past, and we can label this time  $t = 0$ . Since  $a(0) = 0$  at this point, the density  $\rho$  diverges, as does the Hubble expansion parameter. One can see also that, because  $a(t)$  is a concave function, the time between the singularity and the epoch  $t$  must always be less than the characteristic expansion time of the Universe,  $\tau_H = 1/H$ .

# The equations

$$\dot{a}^2(t) + Kc^2 = \frac{8}{3}\pi G\rho(t)a^2(t)$$

$$\frac{K}{a^2(t)} = \frac{1}{c^2} \frac{\dot{a}^2(t)}{a^2(t)} \left( \frac{\rho(t)}{\rho_c(t)} - 1 \right); \quad \rho_c(t) \equiv \frac{3}{8\pi G} \frac{\dot{a}^2(t)}{a^2(t)}$$

$$\Omega_{0w} = \frac{\rho_{0w}}{\rho_c}$$

$$\left(\frac{\dot{a}}{a_0}\right)^2 = H_0^2 \left[ \Omega_{0w} \left(\frac{a_0}{a}\right)^{1+3w} + (1 - \Omega_{0w}) \right]$$

$$H^2(t) = H_0^2 \left(\frac{a_0}{a}\right)^2 \left[ \Omega_{0w} \left(\frac{a_0}{a}\right)^{1+3w} + (1 - \Omega_{0w}) \right]$$

# Flat models

- We shall find the solution to these equations appropriate to a *flat universe*, i.e. with  $\Omega_w = 1$ . When  $w = 0$  this solution is known as *Einstein–de Sitter* universe.
- For  $\Omega_w = 1$ ,

$$\left(\frac{\dot{a}}{a_0}\right)^2 = H_0^2 \left(\frac{a_0}{a}\right)^{1+3w} = H_0^2 (1+z)^{1+3w}$$

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{2/3(1+w)}, \quad t = t_0 (1+z)^{-3(1+w)/2}$$

# Flat universes

## Dust ( $w=0$ )

$$a(t) = a_0 \left( \frac{t}{t_0} \right)^{2/3},$$

$$t = t_0 (1+z)^{-3/2},$$

$$H = \frac{2}{3t} = H_0 (1+z)^{3/2},$$

$$q_0 = \frac{1}{2},$$

$$t_{0m} \equiv t_0 = \frac{2}{3H_0},$$

$$\rho_m = \frac{1}{6\pi G t^2};$$

## Radiation ( $w=1/3$ )

$$a(t) = a_0 \left( \frac{t}{t_0} \right)^{1/2},$$

$$t = t_0 (1+z)^{-2},$$

$$H = \frac{1}{2t} = H_0 (1+z)^2,$$

$$q_0 = 1,$$

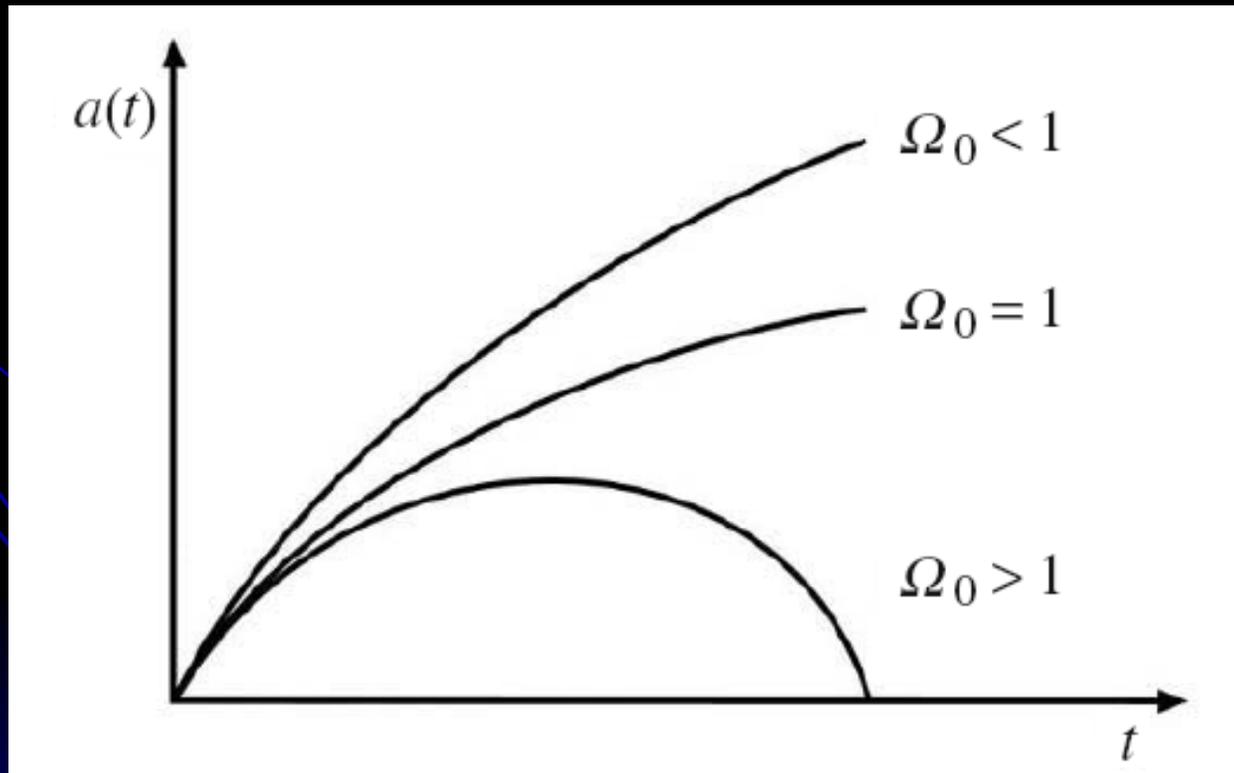
$$t_{0r} \equiv t_0 = \frac{1}{2H_0},$$

$$\rho_r = \frac{3}{32\pi G t^2}.$$

- A general property of flat-universe models is that the expansion parameter  $a$  grows indefinitely with time, with constant deceleration parameter  $q_0$ .
- Increasing  $w$  and, therefore, increasing the pressure causes the deceleration parameter also to increase. Conversely, and paradoxically, a negative value of  $w$  indicating behavior similar to a cosmological constant corresponds to negative pressure (tension) but nevertheless can cause an accelerated expansion.
- Note also the general result that in such models the age of the Universe,  $t_0$ , is closely related to the present value of the Hubble parameter,  $H_0$ .

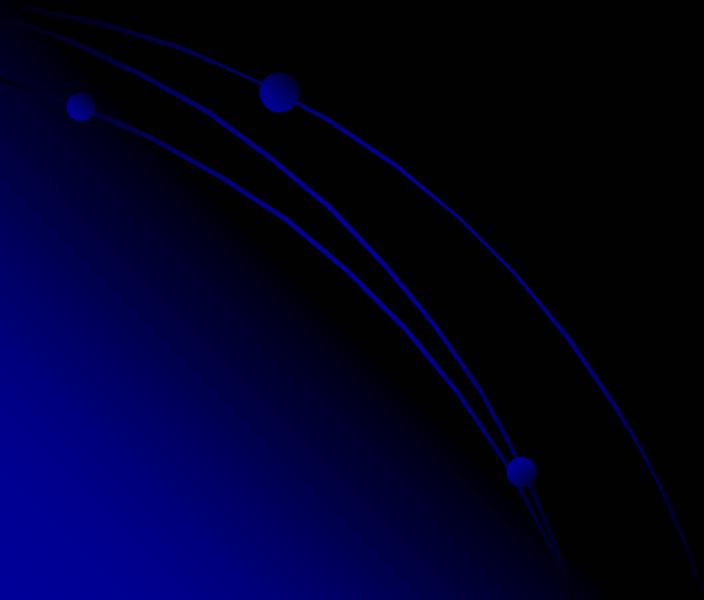
# Curved models

- Evolution of the expansion parameter  $a(t)$  in an open model ( $\Omega_0 < 1$ ), flat or Einstein–de Sitter model ( $\Omega_0 = 1$ ) and closed model ( $\Omega_0 > 1$ ).



# Density parameter

- In expressions derived so far, the quantity that appears is  $\Omega_{w0}$  or, in the special case of  $w = 0$ , just  $\Omega_0$ . This is simply because we have chosen to parametrise the solutions with the value of  $\Omega$  at the time  $t = t_0$ . However, it is very important to bear in mind that  $\Omega$  is a function of time.



- If we instead wish to calculate the density parameter at an arbitrary redshift  $z$ , the relevant expression is

$$\Omega_w(z) = \frac{\rho_w(z)}{[3H^2(z)/8\pi G]} = \frac{\rho_{0w}(1+z)^{3(1+w)}}{[3H^2(z)/8\pi G]}$$

$$H^2(z) = H_0^2(1+z)^2 \left[ \Omega_{0w}(1+z)^{1+3w} + (1 - \Omega_{0w}) \right]$$

$$\Omega_w(z)^{-1} - 1 = \frac{\Omega_{0w}(z)^{-1} - 1}{(1+z)^{1+3w}}$$

- Notice that if  $\Omega_{0w} > 1$ , then  $\Omega_w(z) > 1$  for all  $z$ ; likewise, if  $\Omega_{0w} < 1$ , then  $\Omega_w(z) < 1$  for all  $z$ ; on the other hand, if  $\Omega_{0w} = 1$ , then  $\Omega_w(z) = 1$  for all time. The reason for this is clear: the expansion cannot change the sign of the curvature parameter  $K$ .
- As  $z$  tends to infinity, i.e. as we move closer and closer to the Big Bang,  $\Omega_w(z)$  always tends towards unity.

# Cosmological horizons

- Consider the question of finding the set of points capable of sending light signals that could have been received by an observer up to some generic time  $t$ . For simplicity, place the observer at the origin of our coordinate system  $O$ . The set of points in question can be said to have the possibility of being causally connected with the observer at  $O$  at time  $t$ . It is clear that any light signal received at  $O$  by the time  $t$  must have been emitted by a source at some time  $t'$  contained in the interval between  $t' = 0$  and  $t'$ . The set of points that could have communicated with  $O$  in this way must be inside a sphere centered upon  $O$  with proper radius

$$R_H(t) = a(t) \int_0^t \frac{cdt'}{a(t')}$$

$$R_H(t) = a(t) \int_0^t \frac{cdt'}{a(t')}$$

- Since one takes the lower limit of integration to be zero, there is the possibility that the integral diverges because  $a(t)$  also tends to zero for small  $t$ . In this case the observer at O can, in principle, have received light signals from the whole Universe. If, on the other hand, the integral converges to a finite value with this limit, then the spherical surface with centre O and radius  $R_H$  is called the *particle horizon* at time  $t$  of the observer.

- In this case, the observer cannot possibly have received light signals, at any time in his history, from sources which are situated at proper distances greater than  $R_H(t)$  from him at time  $t$ . The particle horizon thus divides the set of all points into two classes: those which can, in principle, have been observed by  $O$  (inside the horizon), and those which cannot (outside the horizon).

- An approximate expression for particle horizon is

$$R_H(t) \sim \frac{c}{H_0 \sqrt{\Omega_{0w}}} \frac{2}{3w+1} \left( \frac{a}{a_0} \right)^{3(1+w)/2}$$

$$R_H(t) \sim 3 \frac{1+w}{3w+1} ct$$

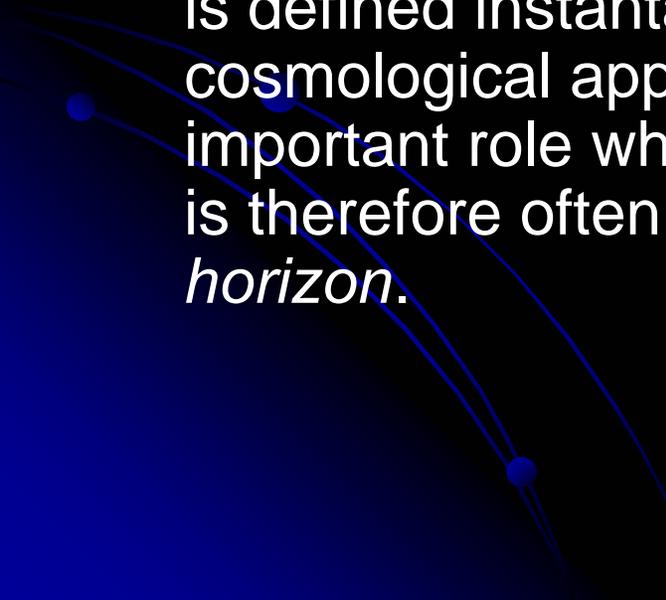
- Which becomes exact for  $\Omega_{0w} = 1$ ; interesting special cases are  $R_H(t) = 3ct$  for the flat dust model and  $R_H(t) = 2ct$  for a flat radiative model.
- In a pure de Sitter cosmological model, which expands exponentially and lasts forever, there is no particle horizon because the integral is not finite.

# Hubble sphere

- One should point out the distinction between the cosmological particle horizon and the *Hubble sphere*, or *speed-of-light sphere*,  $R_c$ . The radius of the Hubble sphere, the Hubble radius, is defined to be the distance from O of an object moving with the cosmological expansion at the velocity of light with respect to O. This can be seen very easily to be

$$R_c = c \frac{a}{\dot{a}} = \frac{c}{H}$$

- One can see that, if  $p > -1/3\rho c^2$ , the value of  $R_c$  coincides, at least to order of magnitude, with the distance to the particle horizon,  $R_H$ . For example, if  $\Omega_w = 1$ , we have  $R_c = 3/2 (1 + w)ct = 1/2 (1 + 3w)R_H \cong R_H$ .
- One can think of  $R_c$  as being the proper distance traveled by light in the characteristic expansion time, or *Hubble time*, of the universe,  $\tau_H = 1/H$ ,

- The Hubble sphere is, however, not the same as the particle horizon. For one thing, it is possible for objects to be outside an observer's Hubble sphere but inside his particle horizon. It is also the case that, once inside an observer's horizon, a point stays within the horizon forever. This is not the case for the Hubble sphere: objects can be within the Hubble sphere at one time  $t$ , outside it sometime later, and, later still, they may enter the sphere again. The key difference is that the particle horizon at time  $t$  takes account of the entire past history of the observer up to the time  $t$ , while the Hubble radius is defined instantaneously at  $t$ . Nevertheless, in some cosmological applications, the Hubble sphere plays an important role which is similar to that of the horizon, and is therefore often called the *effective cosmological horizon*.
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# Models with cosmological constant

- We have already shown how a cosmological constant can be treated as a fluid with equation of state  $p = -\rho c^2$ , i.e. with  $w = -1$ . We know, however, that there is at least some non-relativistic matter and some radiation in the Universe, so a model with only a  $\Lambda$  term can not be complete. In mixed models, with more than one type of fluid and/or contributions from a cosmological constant, the equations describing the evolution become more complicated and closed-form solutions much harder to come by. This is not a problem in the era of fast computers, however, as equivalent results to those of single-fluid cases can be solved by numerical integration.

$$\left(\frac{\dot{a}}{a_0}\right)^2 = H_0^2 \left[ \Omega_{0m} \left(\frac{a}{a_0}\right) + \Omega_{0r} \left(\frac{a}{a_0}\right)^2 + \Omega_{0\Lambda} \left(\frac{a}{a_0}\right)^{-2} + (1 - \Omega_{0m} - \Omega_{0r} - \Omega_{0\Lambda}) \right]$$

