Relativistic Universe

Minkowski space-time and equivalence principle

Events and space-time

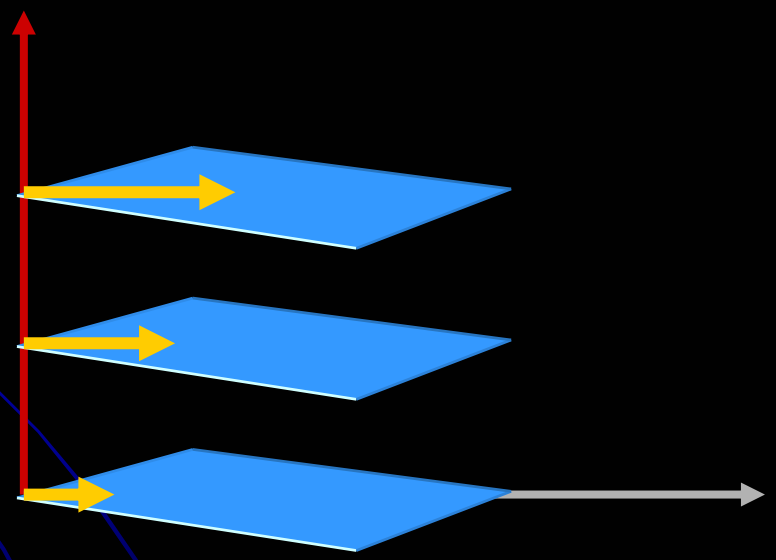
- An *event* is time ``when it happened'' and place ``where it happened''. Thus, an event is the quadruple (ct, x^1, x^2, x^3). In special relativity we can write instead ($x^0 = ct$, x^1, x^2, x^3).
- Space-time is a (4D) set of all events; in principle this set may have some other structures.

Galilean space-time

- There exists a universal time. Galilean space-time is foliated by surfaces of constant time.
- On these surfaces one introduces Euclidean distance (a rotational invariant quadratic form):

  $(x - x')^2 + (y - y')^2 + (z - z')^2$

- Galilean transformations are (linearly time-dependent) shifts on each of these constant time surfaces.



Minkowski space-time

- In Galilean space-time there is no way to compare time and space (because time and space have different physical dimensions and there is no scale to convert one to another.)
- In special relativity we have a scale of velocity, so one can unify space and time.

Doubly special relativity

- In DSR one introduces an additional scale of mass, and, in principle, a unified treatment of the whole phase space is possible.

Invariance of space-time interval

- In system at rest

$$\Delta t = 2d / c, \Delta x = 0$$

- In moving system

$$\Delta t' = \frac{2d / c}{\sqrt{1 - v^2/c^2}},$$

$$\Delta x' = \frac{2d\, v/c}{\sqrt{1 - v^2/c^2}}$$

- Therefore we find

$$\Delta s^2 = \Delta t^2 - \Delta x^2 = \Delta t'^2 - \Delta x'^2$$

- i.e., the interval is invariant!

Minkowski space-time

- Thus Minkowski space-time is a 4 dimensional space with invariant metric (a measure of distance of neighboring events)

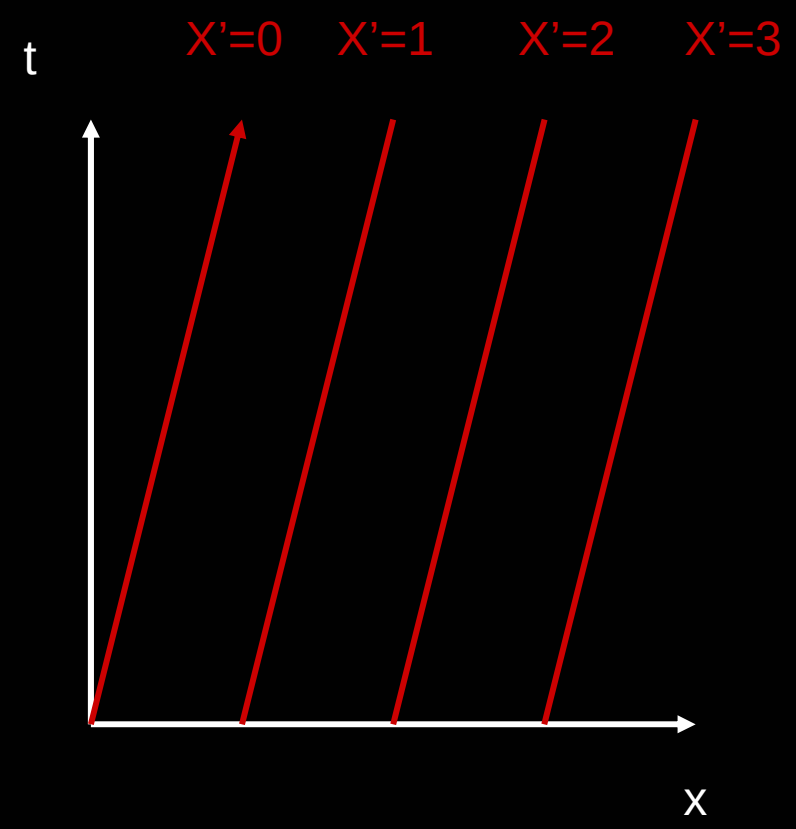
$$ds^2 = dx^{0^2} - dx^{1^2} - dx^{2^2} - dx^{3^2} =$$

$$= g_{\mu\nu} dx^\mu dx^\nu;$$

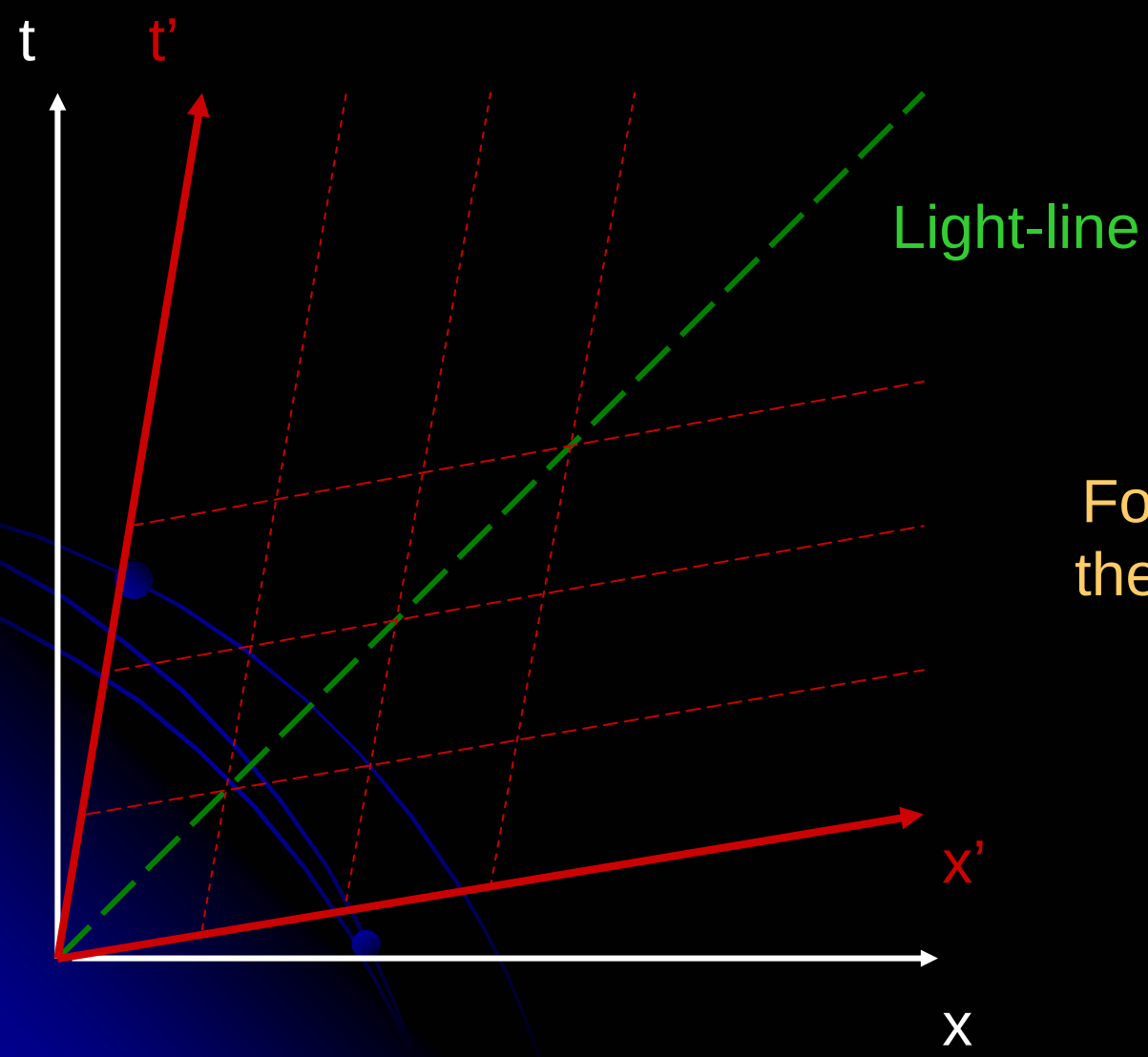
$$g_{\mu\nu} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix}$$

Coordinates in Minkowski s-t

- An observer at rest has coordinates *t, x*; we want to construct coordinates for moving observer.
- Red lines represent points at rest in moving system

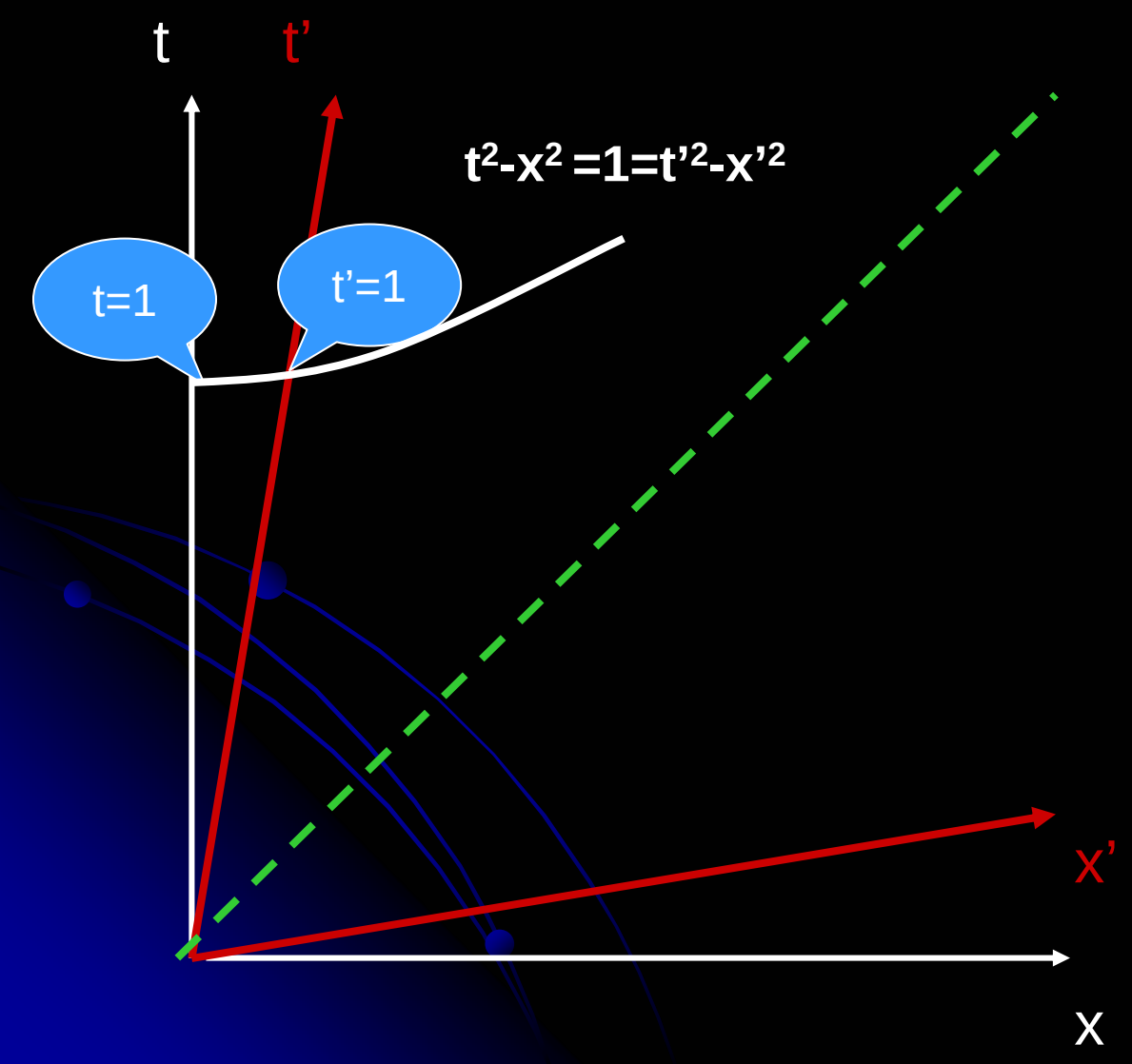


Adding *x'* axis

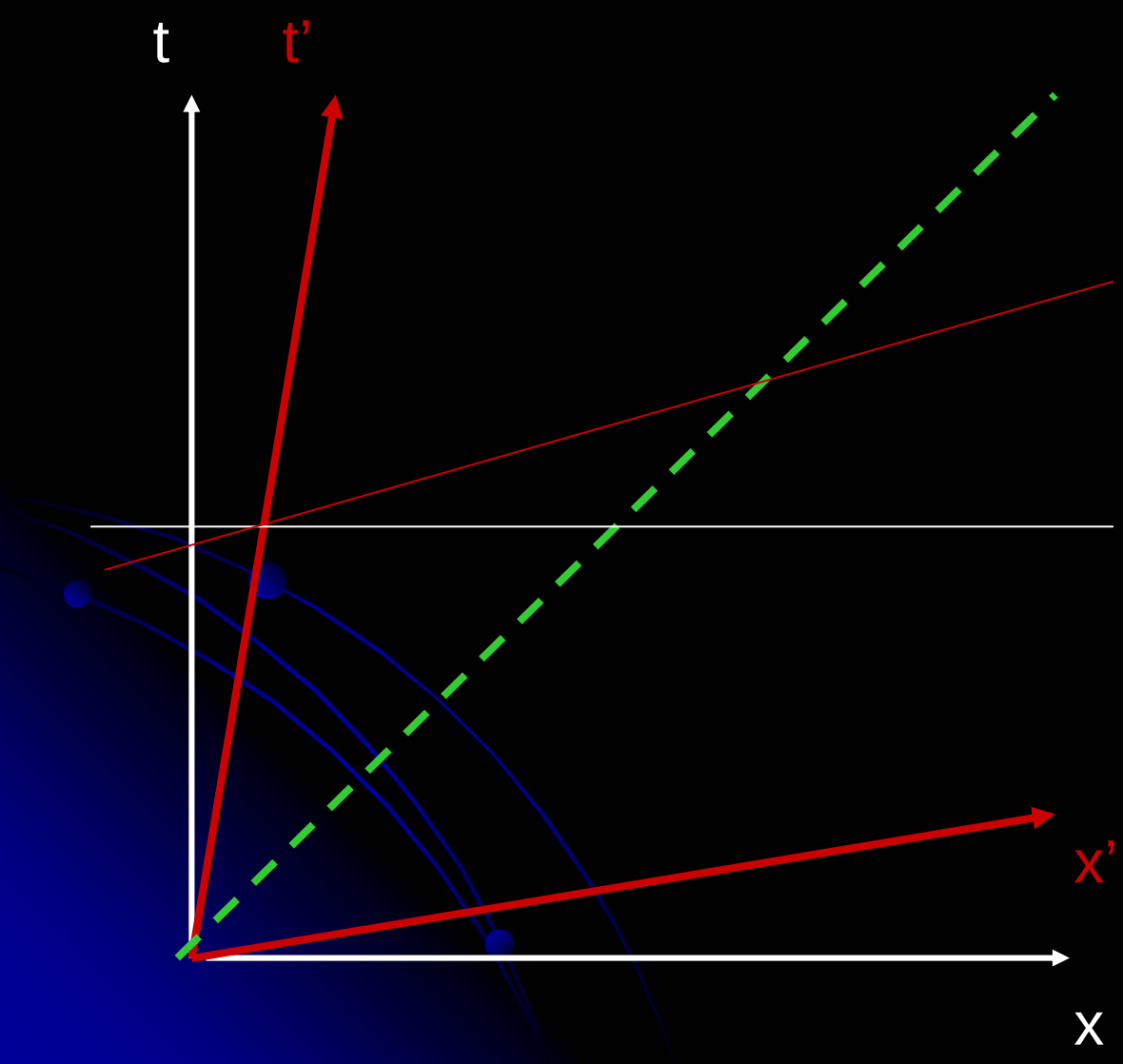


For both observers the speed of light is the same and equal 1

Synchronization



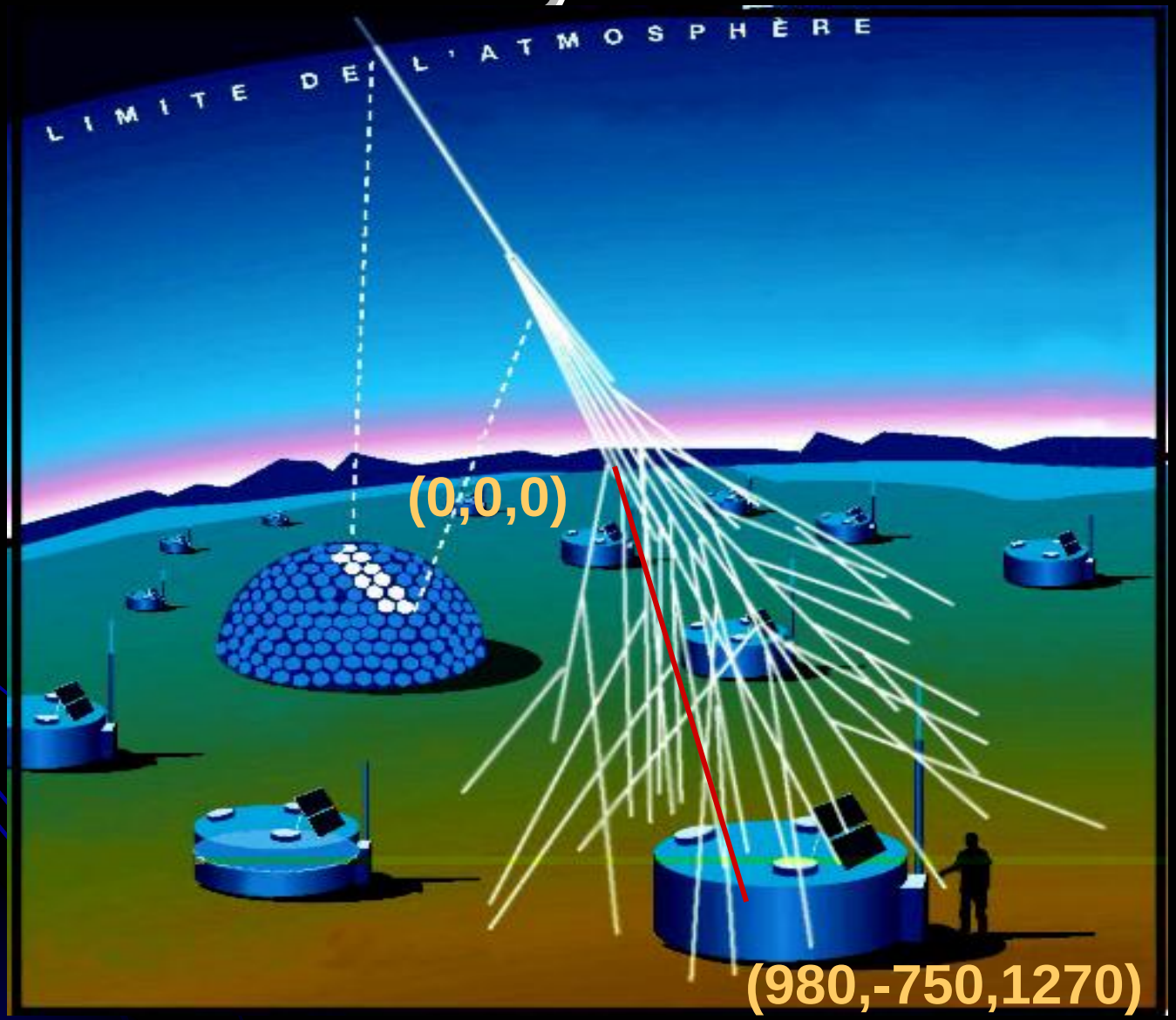
Equal time lines



Equal time line in system x’-t’

Equal time line in system x-t

Cosmic ray muon



Time dilatation

- The distance between two points is

$$D = \sqrt{980^2 + 750^2 + 1270^2} = 1771$$

- The lifetime of muon (in its proper frame) is 2 μs

$$\Delta\tau = ds = \sqrt{2^2 - \frac{1}{300^2}\left(0^2 + 0^2 + 0^2\right)} = 2\mu\sec$$

- Time of flight, as observed from Earth

$$\Delta T = \sqrt{\Delta\tau^2 + D^2} = 6.23\mu\sec$$

Composition of velocities

- Let two observers (x,t), (x',t') move with respect to each other with velocity v along x axis. Let a particle move with velocity u with respect to the first, and u' with respect to the second. What is the relation between u and u'?

- First we know that

$$u = \frac{\Delta x}{\Delta t}, u' = \frac{\Delta x'}{\Delta t'}$$

- And from Lorentz transformation

$$\Delta x = \frac{\Delta x + v\Delta t}{\sqrt{1 - v^2/c^2}}, \quad \Delta t = \frac{\Delta t + v\Delta x/c^2}{\sqrt{1 - v^2/c^2}}$$

$$u = \frac{u' + v}{1 + u'v/c^2}$$

Formal structures of SR

- Let us summarize what we have learned using more formal language this time.
- Events: an event can be arranged into a column

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Minkowski space-time

- Space-time is a set of all events.
- In Minkowski space-time we have additional structure, which is Minkowski metric (interval, distance between neighboring points)

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \quad x + dx = \begin{pmatrix} x^0 + dx^0 \\ x^1 + dx^1 \\ x^2 + dx^2 \\ x^3 + dx^3 \end{pmatrix}$$

Minkowski metric

$$ds^2 = dx^{0^2} - dx^{1^2} - dx^{2^2} - dx^{3^2}$$

$$= \sum_{\mu,\nu=0}^{3} g_{\mu\nu} dx^{\mu} dx^{\nu}$$

$$g_{\mu\nu} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix}$$

Lorentz transformations

- If one inertial observers atributes to an event coordinates *x* and another *x'*, then the coordinates are related by Lorentz transformation

$$(x')^{\mu} = \sum_{\nu=0}^{3} L^{\mu}{}_{\nu}\, x^{\nu}$$

- Lorentz transformations do not change the metric

$$\sum_{\mu,\nu=0}^{3} g_{\mu\nu} dx^{\mu} dx^{\nu} = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} dx'^{\mu} dx'^{\nu}$$

Lorentz matrix

- It can be shown that for the transformation in *z* direction we have

$$L^{\mu}{}_{\nu} = \begin{array}{c} \\ \\ \mu \\ \downarrow \end{array} \begin{array}{c} \nu\rightarrow \\ \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\xi & -\sinh\xi \\ 0 & 0 & -\sinh\xi & \cosh\xi \end{Vmatrix} \end{array}$$

Electromagnetic field tensor

$$F^{\mu\nu} = \begin{array}{c} \\ \mu \downarrow \end{array} \begin{array}{c} \nu \rightarrow \\ \begin{Vmatrix} 0 & B_3 & -B_2 & -E_1 \\ B_3 & 0 & B_1 & -E_2 \\ B_2 & -B_1 & 0 & -E_3 \\ E_1 & E_2 & E_3 & 0 \end{Vmatrix} \end{array},$$

$$(F')^{\mu\nu} = L^{\mu}{}_{\alpha} L^{\nu}{}_{\beta} F^{\alpha\beta}$$

Levi-Civita tensor

$$\varepsilon^{\mu\nu\rho\sigma} = -\varepsilon^{\mu\rho\nu\sigma} = \varepsilon^{\mu\rho\sigma\nu} = \cdots$$

$$\varepsilon^{\mu\nu\rho\sigma} = 0 \,\text{if two indeces are equal}$$

Raising and lowering indices

- Metric tensor can be used to raise and lower indices

$$g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}(1, \quad -1, \quad -1, \quad -1)$$

$$g_{\mu\nu} x^{\mu} y^{\nu} = x^{\mu} y_{\mu} = x_{\mu} y^{\mu}$$

$$F^{\mu\rho} g_{\rho\nu} = F^{\mu}{}_{\nu}$$

Derivatives

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}, \quad \partial_\mu x^\nu = \delta^\mu{}_\nu$$

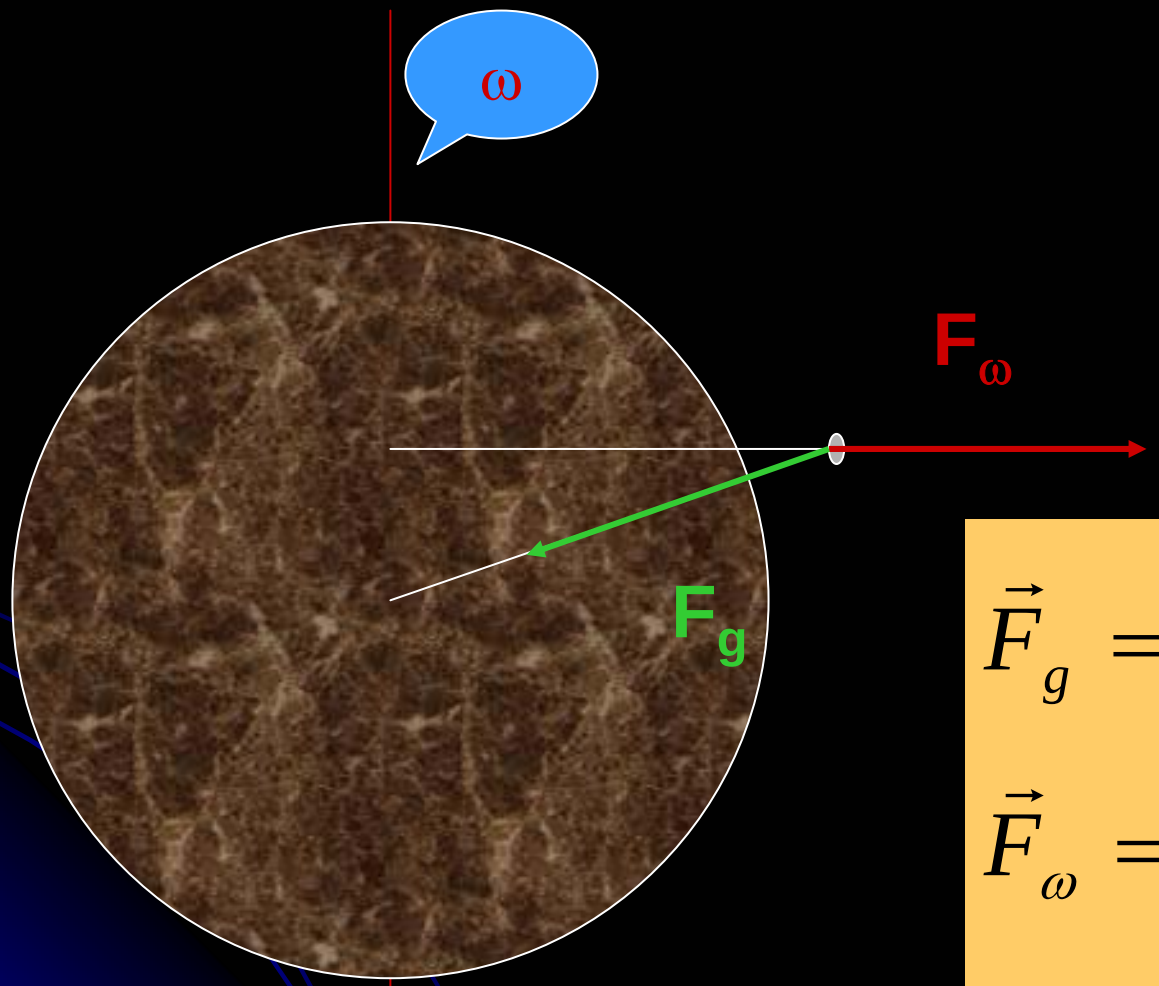
$$\partial^\alpha = g^{\alpha\mu}\partial_\mu, \; \partial^\alpha x^\nu = g^{\alpha\nu} \qquad \cdots$$

Vacuum Maxwell equations

$$\partial_\mu F^{\mu\nu} = 0$$

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F^{\rho\sigma} = 0$$

Equivalence principle



$$\vec{F}_g = -GM_\oplus m_g \frac{\vec{r}}{r^3}$$

$$\vec{F}_\omega = m_i \omega^2 \vec{r}_a,$$

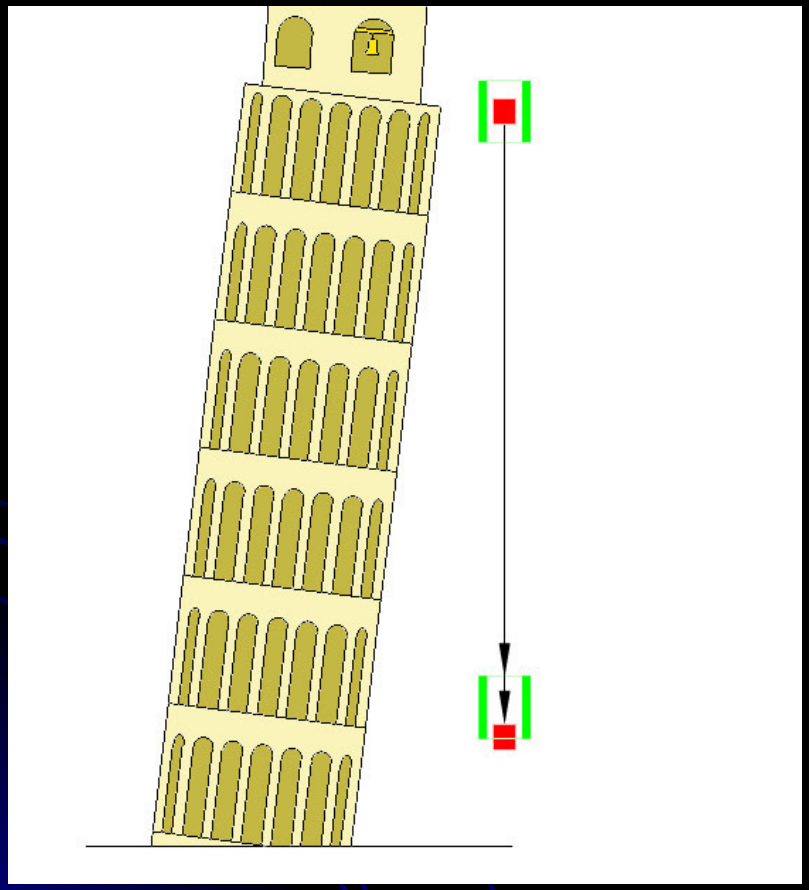
$$\vec{r}_a = \vec{r} - \frac{(\vec{r}\,\vec{\omega})\vec{\omega}}{\omega^2}$$

- Suppose we have two bodies of different composition. Then if the ratio of gravitational and inertial mass is not the same, the total forces acting on these bodies will not be the same and the difference can be measured in principle.

Eotvos experiment

- The experiment has been made with great accuracy (10^{-8}) by baron Eotvos in Budapest in 1922 using torsion balance.

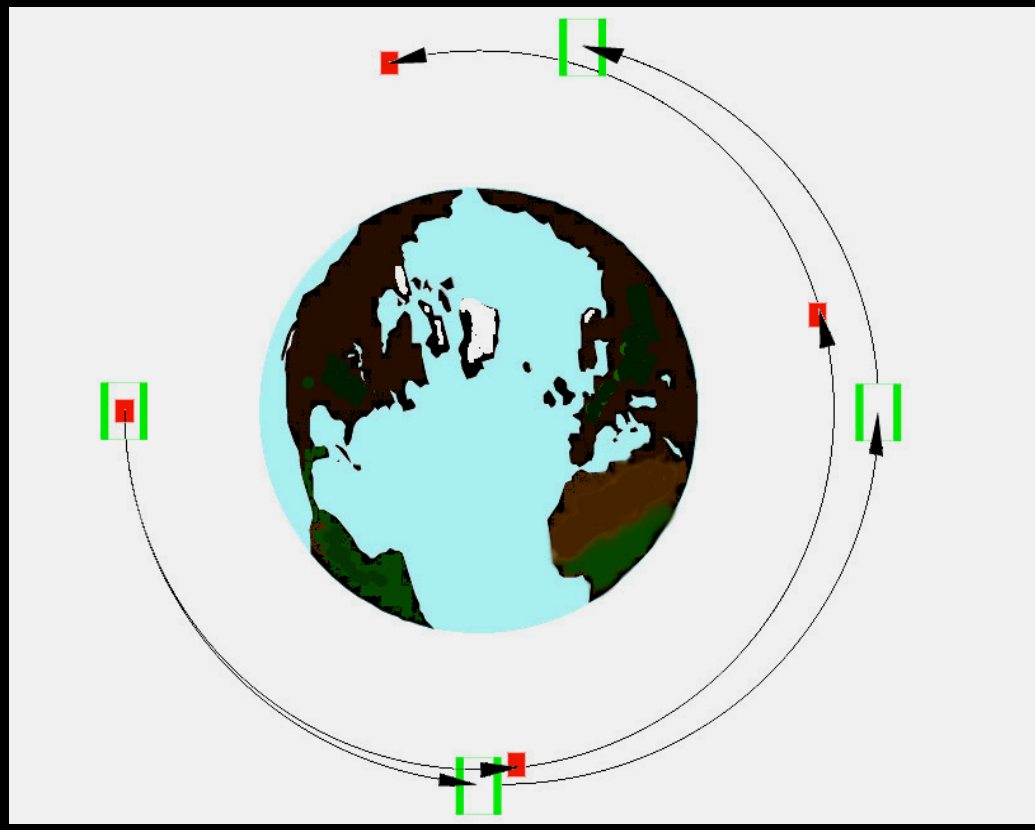
Free fall experiment



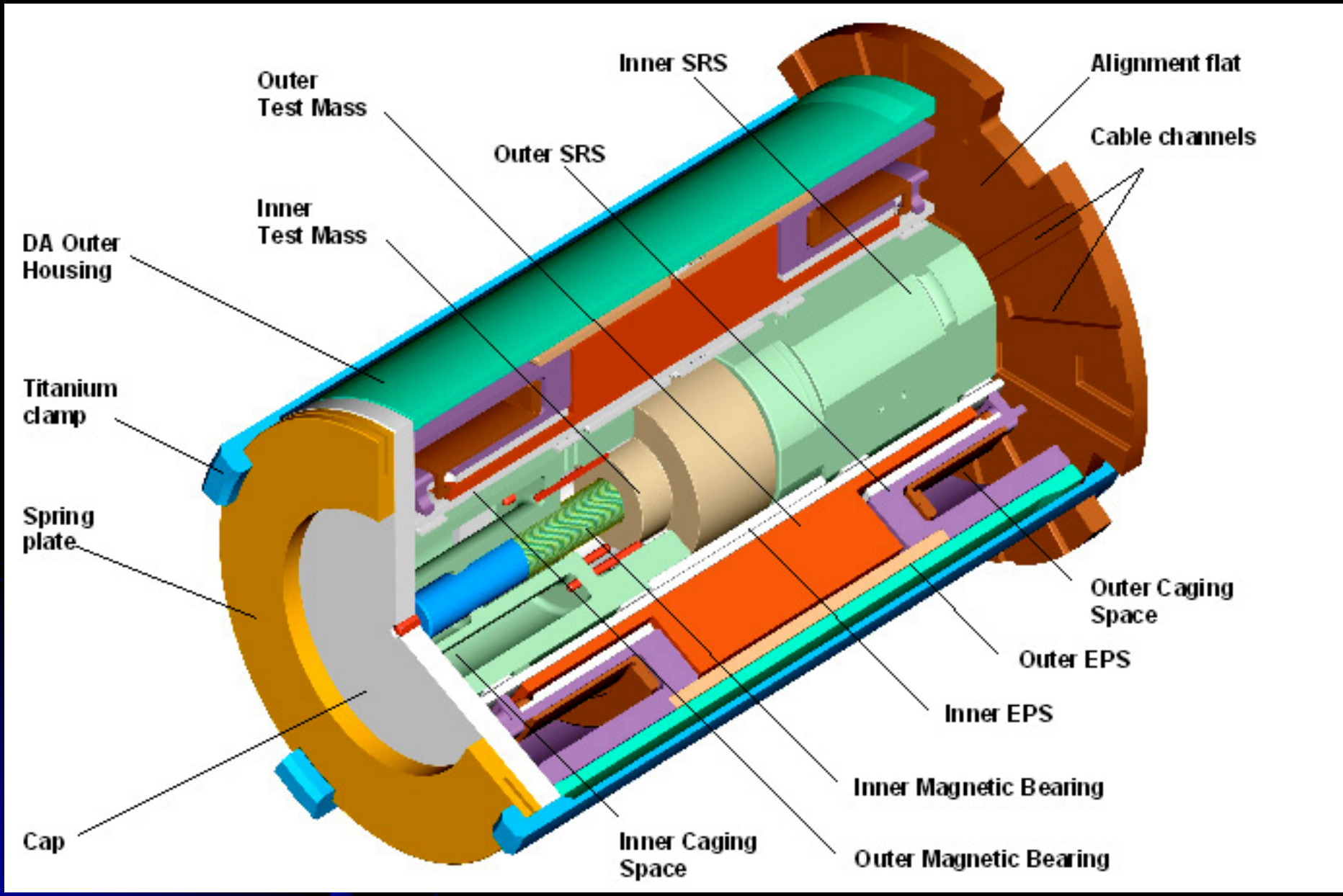
Galileo, 1590; accuracy 10^{-2}

STEP satellite

In STEP, the masses are in free fall in an orbit around the Earth, and if there is a violation of the Equivalence Principle they tend to follow slightly different orbits



Accuracy 10^{-18}


Inner SRS
Alignment flat
Outer
Test Mass
Cable channels
Outer SRS
Inner
Test Mass
DA Outer
Housing
Titanium
clamp
Spring
plate
Outer Caging
Space
Outer EPS
Inner EPS
Inner Magnetic Bearing
Cap
Inner Caging
Space
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http://einstein.stanford.edu/STEP/