Relativistic Universe

General relativity I:

Rindler space

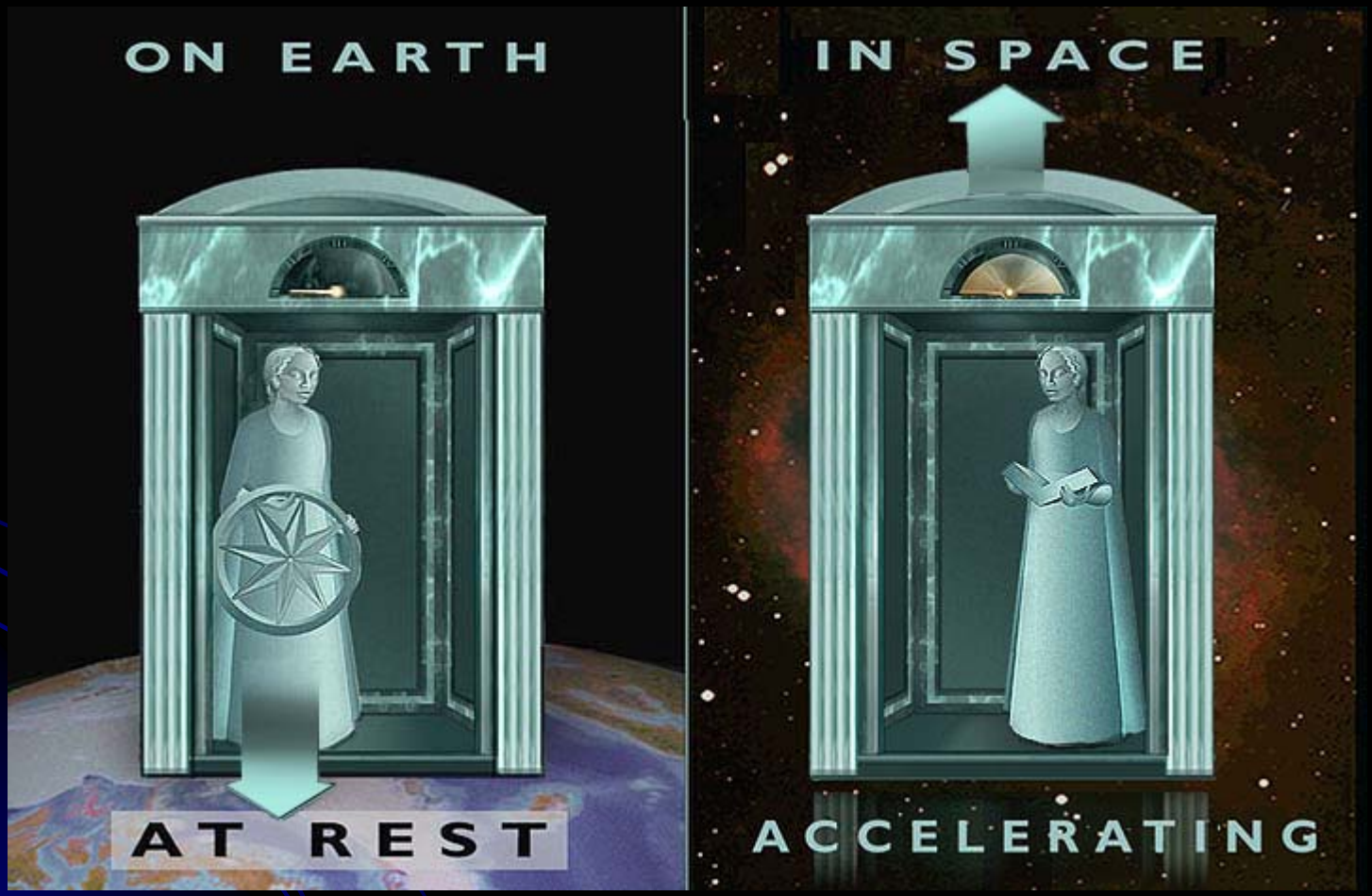
GR in a nutshell

Relativity Principle, which says that *locally* one can describe physical phenomena in language of Special Relativity; and

Equivalence Principle, which says that Effects of accelerations and gravity are indistinguishable.

Einstein elevator

- Imagine two elevators, one at rest of the Earth's surface, one accelerating in space.


ON EARTH
AT REST
IN SPACE
ACCELERATING

- To an observer inside the elevator (no windows) there is no physical experiment that he/she could perform to differentiate between the two scenarios.
- And by the same token, if the observer is placed in the free-falling elevator he/she cannot detect gravitational field by any (local) experiment

- Thus:

1. Gravitational field = (local) acceleration;
2. Gravitational field can be identified with geometry of space-time

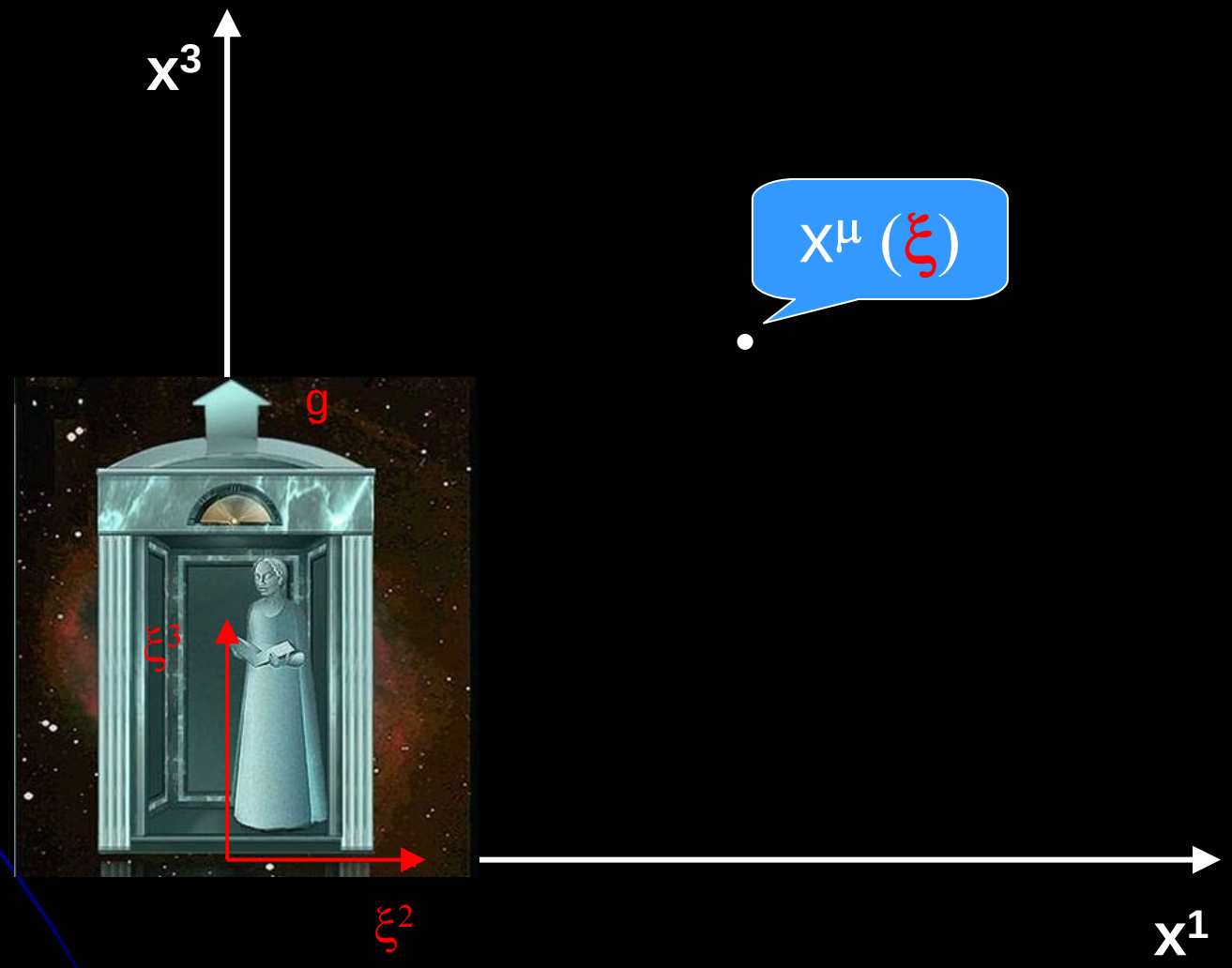
Rindler space

- To get some idea about how to describe gravitational field, let us consider constantly accelerating observer. According to the equivalence principle he/she will feel gravitational field.

Accelerating observer

Time as measured by inertial observer is t;

Time as measured by accelerating observer is τ

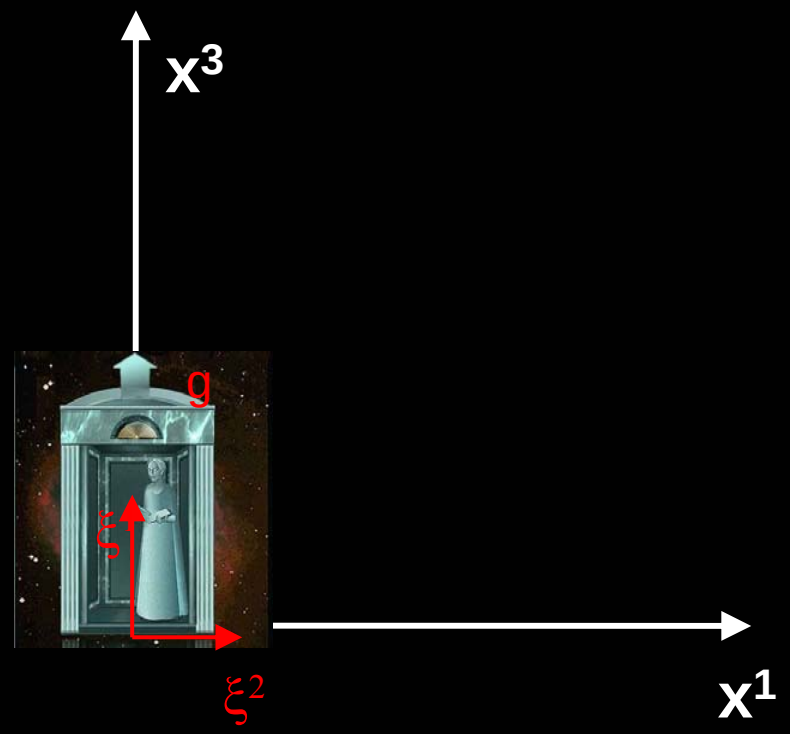


At τ=0

$$\vec{g} = \begin{pmatrix} 0, & 0, & g \end{pmatrix}$$

$$\vec{v} = 0$$

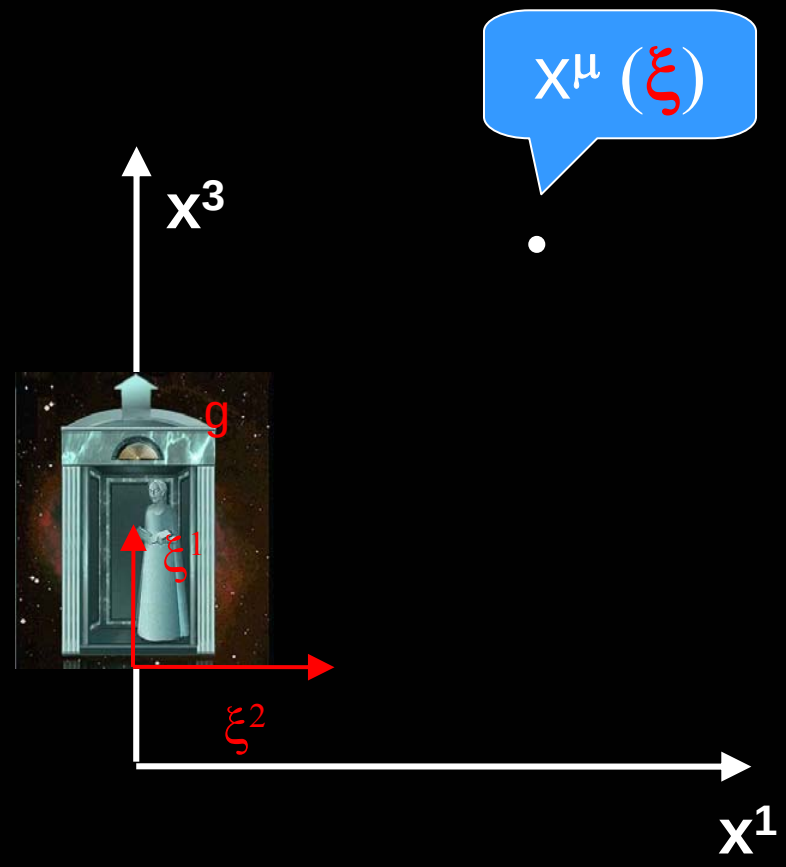
$$\begin{pmatrix} \vec{x}(\vec{\xi},0) \\ 0 \end{pmatrix} = \begin{pmatrix} \vec{\xi} \\ 0 \end{pmatrix}$$



After dτ passed

$$\vec{v} = \vec{g}\, d\tau$$

$$\begin{pmatrix} \vec{x}(\vec{\xi}, d\tau) \\ t(\vec{\xi}, d\tau) \end{pmatrix} = \begin{pmatrix} \vec{\xi} \\ d\tau \end{pmatrix}$$



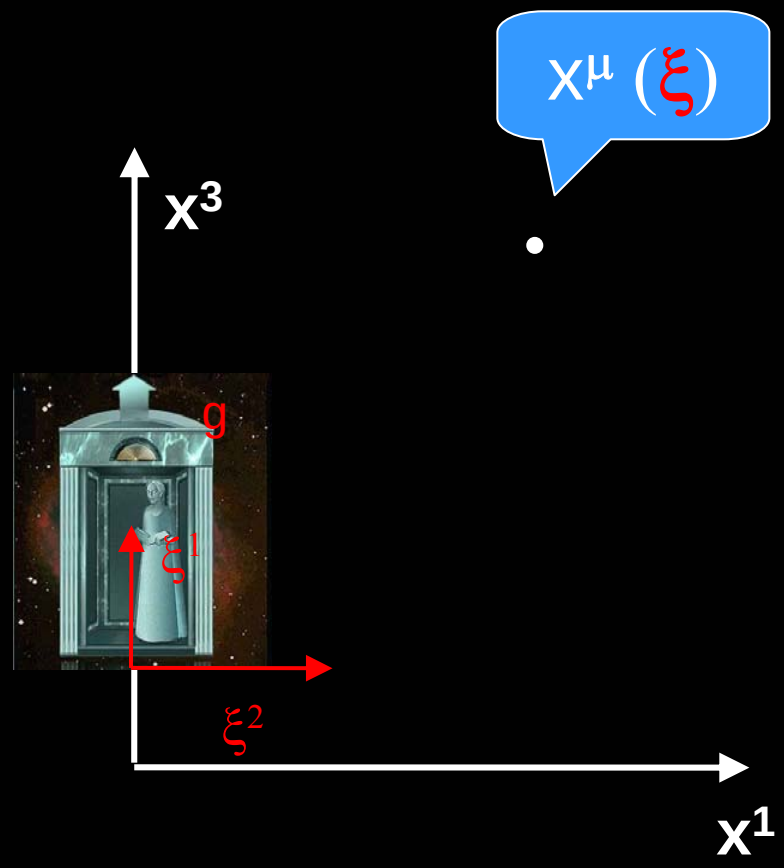
Lorentz transformation

- But now the moving observer moves with velocity v, so he/she will see all points Lorentz transformed.

- Since the elapsed time $d\tau$ is infinitesimal, the Lorentz transformation will differ from identity only by infinitesimal amount

$$1+\delta L=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -gd\tau \\ 0 & 0 & gd\tau & 1 \end{pmatrix}$$

- So now the coordinates of a point in two systems are related by Poincare transformation



$$\begin{pmatrix} \vec{x} \\ t \end{pmatrix} = (1+\delta L)\begin{pmatrix} \vec{\xi} \\ 0 \end{pmatrix} + \begin{pmatrix} \vec{0} \\ d\tau \end{pmatrix}$$

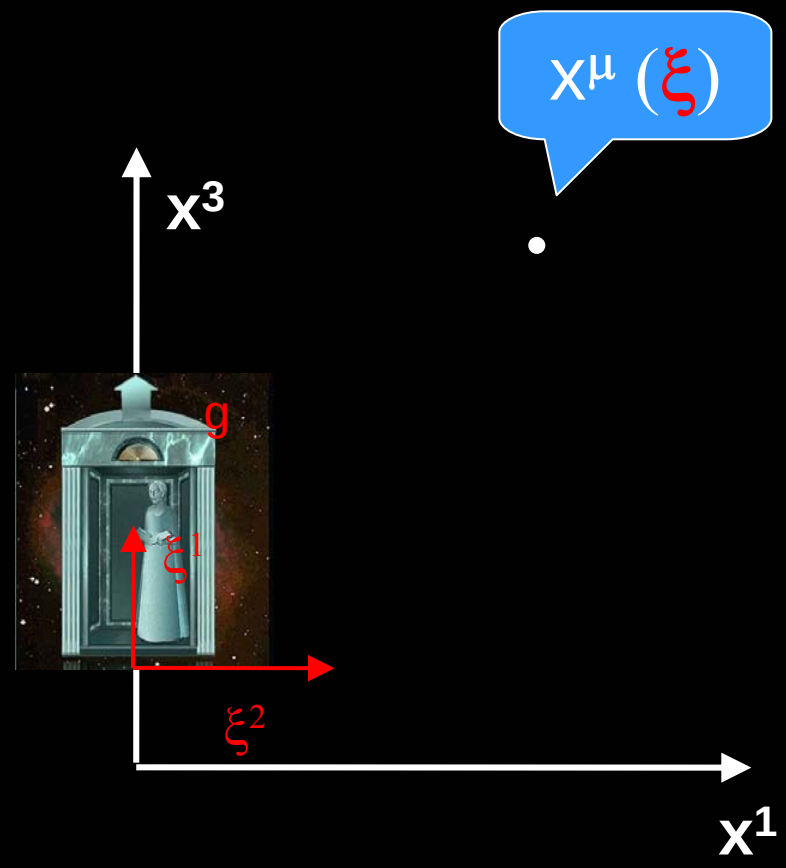
- But it follows that

$$\delta L\begin{pmatrix} 0 \\ 0 \\ 1/g \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -gd\tau \\ 0 & 0 & gd\tau & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1/g \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ d\tau \end{pmatrix}$$

$$\begin{pmatrix} \vec{0} \\ d\tau \end{pmatrix} = \delta L\begin{pmatrix} \vec{g}/g^2 \\ 0 \end{pmatrix}$$

- So that we finally have

$$\begin{pmatrix} \vec{x} + \vec{g}/g^2 \\ t \end{pmatrix} = (1+\delta L) \begin{pmatrix} \vec{\xi} + \vec{g}/g^2 \\ 0 \end{pmatrix}$$



Now after time $\tau = N d\tau$

- Since the acceleration is constant we obtain new positions in terms of the old ones always by means of the same Lorentz transformation. Thus

$$\begin{pmatrix} \vec{x} + \vec{g}/g^2 \\ t \end{pmatrix} = (1+\delta L)^N \begin{pmatrix} \vec{\xi} + \vec{g}/g^2 \\ 0 \end{pmatrix}$$

$$\tau = N d\tau;\ L(\tau) = (1+\delta L)^N;\ L(\tau + d\tau) = (1+\delta L)L(\tau)$$

$$\delta L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -g \\ 0 & 0 & g & 0 \end{pmatrix} d\tau; \quad L(\tau) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & A(\tau) & -B(\tau) \\ 0 & 0 & B(\tau) & A(\tau) \end{pmatrix}$$

$$L(0) = 1; \quad {dA}/{d\tau} = gB; \quad {dB}/{d\tau} = gA;$$

$$A(\tau) = \cosh(g\tau); B(\tau) = \sinh(g\tau)$$

So finally …

$$x^{\mu}(\vec{\xi},\tau) = \begin{pmatrix} \xi^1 \\ \xi^2 \\ \cosh(g\tau)(\xi^3 + 1/g) - 1/g \\ \sinh(g\tau)(\xi^3 + 1/g) \end{pmatrix}$$

Rindler space in $x^3 - t$ plane

- We can consider only the $z - t$ plane, where

$$\begin{pmatrix} z \\ t \end{pmatrix}(\zeta, \tau) = \begin{pmatrix} \cosh(g\tau)(\zeta + 1/g) - 1/g \\ \sinh(g\tau)(\zeta + 1/g) \end{pmatrix}$$

Non-relativistic limit

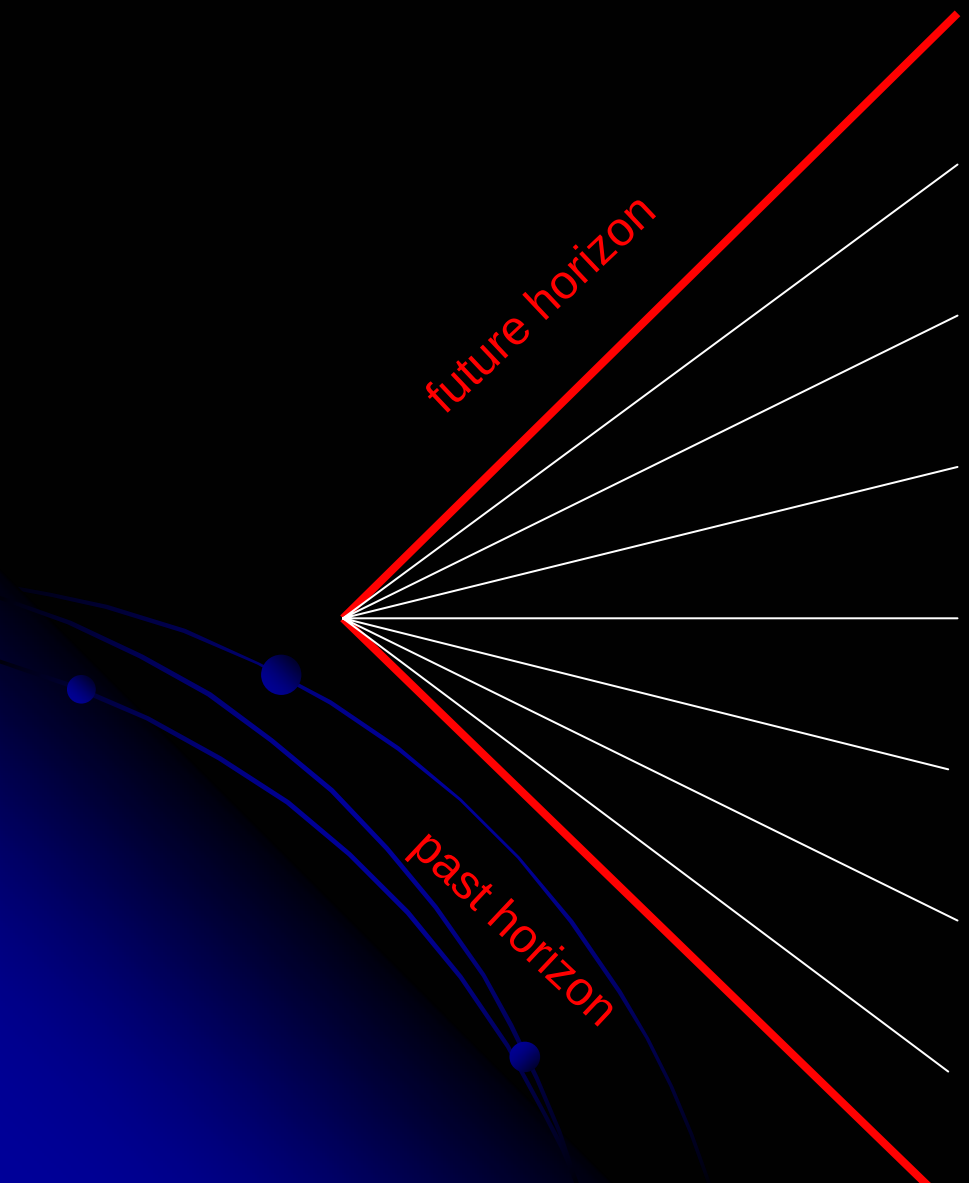
- Let us reintroduce *c:*

$$\begin{pmatrix} z \\ t \end{pmatrix}(\zeta,\tau) = \begin{pmatrix} \cosh\left(\frac{g\tau}{c}\right)(\zeta + c^2/g) - c^2/g \\ \sinh\left(\frac{g\tau}{c}\right)(\zeta/c + c/g) \end{pmatrix}$$

- In the non-relativistic limit $g\tau/c$ is much smaller than 1, so expanding to 2nd order

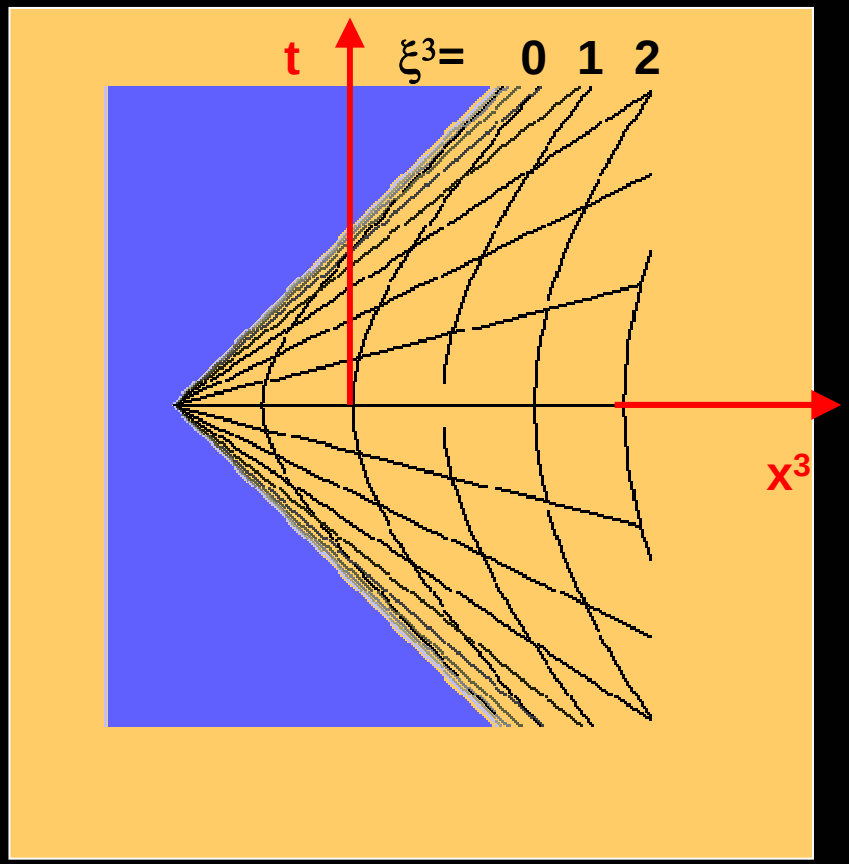
$$\begin{pmatrix} z \\ t \end{pmatrix}(\zeta,\tau) = \begin{pmatrix} \zeta + \frac{g\tau^2}{2} \\ \tau \end{pmatrix}$$

Egual time lines



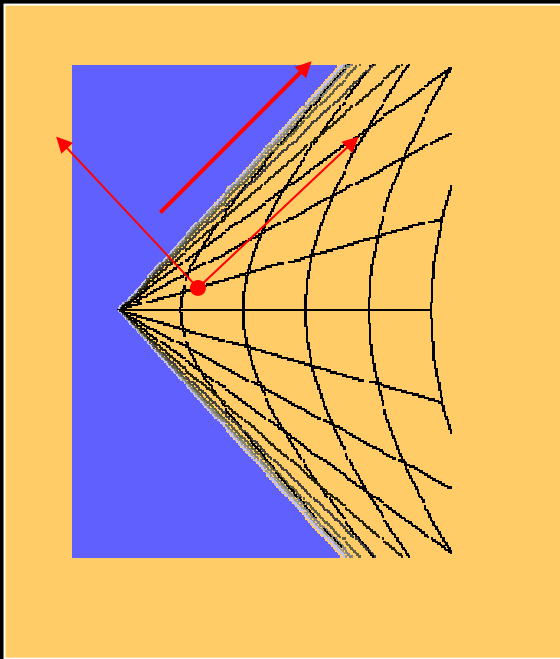
The slope of constant time lines changes from -1 ($\tau=-\infty$) to 1 ($\tau=\infty$)

- The constant ξ^3 lines have future and past horizons as asymptotes. The diagram is therefore of the form
- Rindler space is the quadrant of space-time in ξ coordinates



Properties of Rindler space

- Photon emitted outside the horizon cannot reach the observer in Rindler space. This leads to thermal effect: an accelerating observer sees himself as in the thermal bath of temperature proportional to his acceleration (Unruh law)



Local clock speed

- Suppose it takes $\Delta\tau$ for the accellerating clock to make one tick. We have therefore

$$\Delta t^2 - \Delta z^2 = \Delta\tau^2 - \cancel{\Delta\zeta^2}$$

- Then 1 tick at $\zeta=0$ equals

V(ζ)

$$\rho = \sqrt{\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dx}{d\tau}\right)^2} = (1 + g\zeta)$$

- The rate of clock’s ticking depends on gravitational potential! But this rate is related to space-time distance, so that one can claim that gravitational field is also related to geometry of space-time. Let us check this explicitly.

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

$$dx^\mu = \frac{\partial x^\mu}{\partial \xi^\alpha} d\xi^\alpha;$$

$$dx^3 = g\sinh(g\tau)(\xi^3 + 1/g)d\tau + \cosh(g\tau)d\xi^3$$

$$dt = g\cosh(g\tau)(\xi^3 + 1/g)d\tau + \sinh(g\tau)d\xi^3$$

$$dx^1 = d\xi^1; dx^2 = d\xi^2$$

Metric of Rindler space

$$ds^2 = (1 + g\xi^3)^2 d\tau^2 - d\xi^{1^2} - d\xi^{2^2} - d\xi^{3^2}$$

$$ds^2 = g_{\mu\nu} d\xi^\mu d\xi^\nu;$$

$$g_{00} = 1 + 2V(\xi)$$

Conclusions

- We can describe gravitational field in terms of (curved) metric on space-time.
- In the weak field limit the gravitational potential, $V=g_{00}-1$.
- The proper time depends on gravitational field:

$$ds = \sqrt{g_{00}}\, d\tau$$