Relativistic Universe

General relativity III:

Motion in central field

From the last lecture we know …

- The equation of motion of test particle in gravitational field

$$g_{\alpha\mu}\frac{d^2x^\mu}{d\lambda^2}+\frac{1}{2}\left(\frac{\partial g_{\alpha\mu}}{\partial x^\nu}+\frac{\partial g_{\alpha\nu}}{\partial x^\mu}-\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}=0$$

- It is convenient to rewrite it in equivalent form

$$\frac{d}{d\lambda}\left(g_{\alpha\mu}\frac{dx^\mu}{d\lambda}\right)=\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$$

Before we proceed …

- Let us consider the case of light-like objects.
- For light the interval is zero in special relativity, and we assume that in general relativity as well.

$$0 = g_{\mu\nu} dx^{\mu} dx^{\nu}$$

- Therefore for light the path $x^{\mu}(\lambda)$ is given by

$$0 = g_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda}$$

The flat space limit

- In flat space $g_{\mu\nu} = \eta_{\mu\nu} = const$ and thus from

$$g_{\alpha\mu}\frac{d^2x^\mu}{d\lambda^2} + \frac{1}{2}\left(\frac{\partial g_{\alpha\mu}}{\partial x^\nu} + \frac{\partial g_{\alpha\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$$

- it follows

$$g_{\alpha\mu}\frac{d^2x^\mu}{d\lambda^2} = 0$$

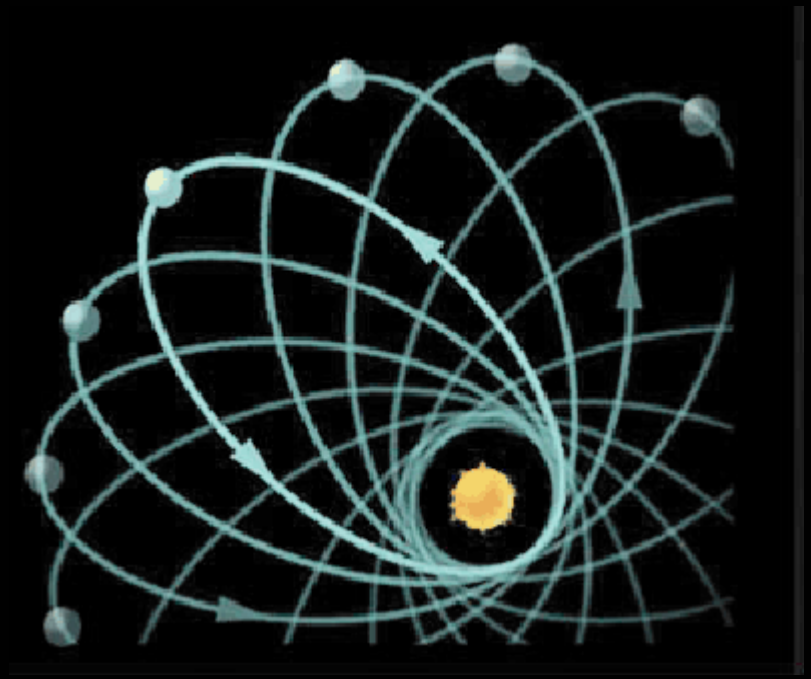
- so that

$$x^\mu(\lambda) = a^\mu + b^\mu\lambda$$

- i.e., the streight line as it should be

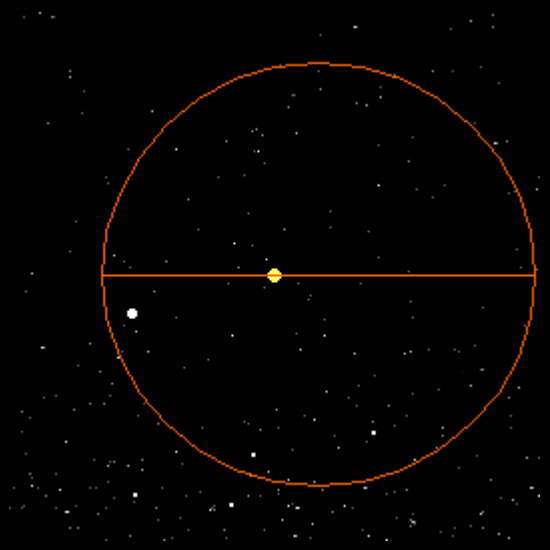
Motion in central field

- We are looking for explanation of the perihelion precession effect.



Perihelion precession

- This effect can be also visualized as follows



To analize this problem …

- We should first know the form of the metric. The metric, corresponding to a single mass, at *r=0* is as follows

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{2m}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1-\frac{2m}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

The interval

- Or the interval of related to this metric (Schwartzschild solution) is of the form

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2m}{r}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Weak field limit …

- This metric is a solution of Einstein equations. We will prove this later, but let us check if this metric has the right weak filed/low velocity limit.
- First notice that we used a convenient set of units, in which G=1 (Newton's constant), c=1 (speed of light). Let us now reintroduce these constants.

Reintroducing G and c

- *G* has dimension $m/s^2 \times m^2/kg = m^3/kg\ s^2$
- *c* has dimension *m/s*
- Thus Gm/rc^2 is dimensionless; thus

$$ds^2 = -\left(1 - \frac{2Gm}{rc^2}\right)c^2 dt^2 + \frac{dr^2}{\left(1 - \frac{2Gm}{rc^2}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

- The first term is right, as it provides the correction to the flat metric due to the gravitational potential.

$$\frac{dr^2}{\left(1-\frac{2Gm}{rc^2}\right)} \approx \left(1+\frac{2Gm}{rc^2}\right)dr^2,$$

$$\frac{2Gm}{rc^2}dr^2 << \frac{2Gm}{rc^2}c^2dt^2$$

Returning to geodesics

- The equation to solve is

$$\frac{d}{d\lambda}\left(g_{\alpha\mu}\frac{dx^{\mu}}{d\lambda}\right)=\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}$$

- with

$$g_{\mu\nu}=\begin{pmatrix} -\left(1-\frac{2m}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1-\frac{2m}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^{2} & 0 \\ 0 & 0 & 0 & r^{2}\sin^{2}\theta \end{pmatrix}$$

In components

- The fact that the metric is diagonal simplifies things greatly

$$\frac{d}{d\lambda}\left(g_{tt}\frac{dt}{d\lambda}\right)=\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial t}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}=0$$

$$\frac{d}{d\lambda}\left(g_{\theta\theta}\frac{d\theta}{d\lambda}\right)=\frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial \theta}\frac{d\phi}{d\lambda}\frac{d\phi}{d\lambda}$$

$$\frac{d}{d\lambda}\left(g_{\phi\phi}\frac{d\phi}{d\lambda}\right)=\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial \phi}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}=0$$

The remaining equation

$$\frac{d}{d\lambda}\left(g_{rr}\frac{dr}{d\lambda}\right)=\frac{1}{2}\left(\frac{\partial g_{tt}}{\partial r}\frac{dt}{d\lambda}\frac{dt}{d\lambda}+\frac{\partial g_{rr}}{\partial r}\frac{dr}{d\lambda}\frac{dr}{d\lambda}+\cancel{\frac{\partial g_{\theta\theta}}{\partial r}\frac{d\theta}{d\lambda}\frac{d\theta}{d\lambda}}+\frac{\partial g_{\phi\phi}}{\partial r}\frac{d\phi}{d\lambda}\frac{d\phi}{d\lambda}\right)$$

Planar motion

$$\frac{d}{d\lambda}\left(g_{\theta\theta}\frac{d\theta}{d\lambda}\right)=\frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial\theta}\frac{d\phi}{d\lambda}\frac{d\phi}{d\lambda}=r^2\sin\theta\cos\theta\frac{d\phi}{d\lambda}\frac{d\phi}{d\lambda}$$

- It follows from this equation that we can consistently take $\theta = const = \pi/2$

ϕ–ϕ equation

- For $\theta = const = \pi / 2$ we have

$$\frac{d}{d\lambda}\left(r^2 \sin^2\theta \frac{d\phi}{d\lambda}\right) = 0 \Rightarrow$$

$$r^2 \dot{\phi} = L = const$$

- (Almost) exactly as in the Newtonian case (where we have time derivative instead of the λ one.)

t-t equation

- Similarly we have

$$\frac{d}{d\lambda}\left(g_{tt}\frac{dt}{d\lambda}\right)=0\Rightarrow$$

$$\left(1-\frac{2m}{r}\right)\frac{dt}{d\lambda}=K=const$$

Simplification

- Remember that

$$g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}=const$$

- By scaling $\lambda \rightarrow \lambda' = a\lambda$ we can make this constant equal +1, -1, or 0 (spacelike, timelike, or light-like)

$$g_{\mu\nu}\frac{dx^{\mu}}{d\lambda'}\frac{dx^{\nu}}{d\lambda'}=a^{2}const$$

$$g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}=-1$$

The final set of equations is

$$\left(1-\frac{2Gm}{r}\right)\dot{t}^2-\left(1-\frac{2Gm}{r}\right)^{-1}\dot{r}^2-r^2\dot{\phi}^2=1$$

$$\left(1-\frac{2Gm}{r}\right)\dot{t}=K$$

$$r^2\dot{\phi}=L$$

- Notice now that we are interested in dependence $r(\phi)$.

$$\frac{dr}{d\phi} = \frac{\dot{r}}{\dot{\phi}}$$

For light-like

- And we have

$$\left(\frac{L}{r^2}\frac{dr}{d\phi}\right)^2 + \left(\frac{L^2}{r^2} + \cancel{1}\right)\left(1 - \frac{2Gm}{r}\right) = K^2$$

Final equation

- We introduce new variables

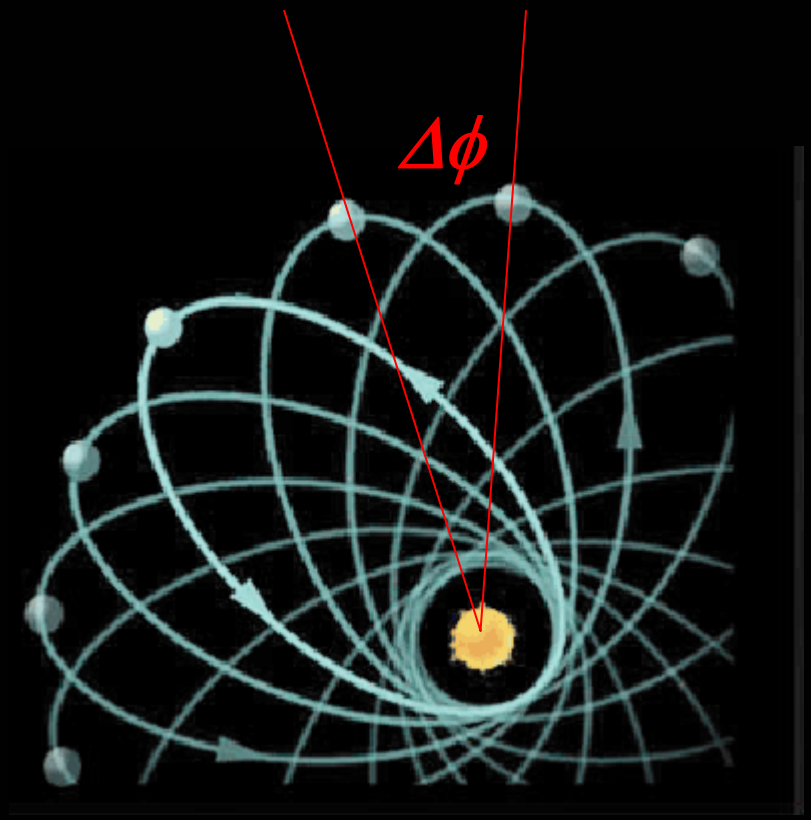
$$w = \frac{Gm}{r}, \quad J = \frac{L}{Gm}$$

- In terms of these variables the equation is

$$\left(\frac{dw}{d\phi}\right)^2 + \frac{1}{J^2}(1-2w)(J^2w^2+1) = \frac{K^2}{J^2}$$

Precession …

- The angle $\Delta\phi$ is small
- If the trajectory is closed then, after one revolution, the radius returns to its initial value $r(\phi+2\pi)=r(\phi)$
- If there is precession, $r(\phi)= r(\phi+2\pi+\Delta\phi)$
- Of course the same holds for w variable



Effective potential

- Our equation for *w* is of the form ``kinetic energy'' + ``potential energy'' = ``total energy''

$$\left(\frac{dw}{d\phi}\right)^2 + \frac{1}{J^2}(1-2w)(J^2w^2+1) = \frac{K^2}{J^2}$$

$$\left(\frac{dw}{d\phi}\right)^2 + V(w) = \frac{K^2}{J^2},$$

$$V(w) = \frac{1}{J^2}(1-2w)(J^2w^2+1)$$

Circular orbit

- Circular orbit, for which $w_c = const$ is possible only if there exist J such that w_c is a minimum of the potential

$$\left.\frac{d}{dw}(1-2w)(J^2w^2+1)\right|_{w=w_c} = -6J^2{w_c}^2 + 2J^2w_c - 1 = 0,$$

$$\left.\frac{d^2}{dw^2}(1-2w)(J^2w^2+1)\right|_{w=w_c} = -12J^2w_c + 2J^2 > 0$$

- The solution is

$$w_c = \frac{1}{6}\left(1 - \sqrt{1 - \frac{12}{J^2}}\right)$$

- And the corresponding energy

$$(1 - 2w_c)(J^2 w_c^{\ 2} + 1) = K_c^{\ 2}$$

Precession

- Now we consider an orbit close to the circular one, with the same angular momentum J, but possibly different energy K

$$\left(\frac{dw}{d\phi}\right)^2 + \frac{1}{J^2}(1-2w)(J^2w^2+1) = \frac{K^2}{J^2}$$

$$(1-2w_c)(J^2{w_c}^2+1) = {K_c}^2$$

- Subtracting we get

$$\left(\frac{dw}{d\phi}\right)^2 + (1-6w_c)(w-w_c)^2 - \cancel{2(w-w_c)^3} = \frac{K^2-{K_c}^2}{J^2}$$

But now we have just an oscillator!

- So we can solve the equation

$$w - w_c = \sqrt{\frac{K^2 - K_c^{\,2}}{J^2}} \cos(\Omega\phi + \phi_0)$$

- with

$$\Omega = \sqrt{1 - 6w_c}$$

- The planet returns to its initial r after ϕ changes by

$$\frac{2\pi}{\sqrt{1 - 6w_c}}$$

So the precession is

$$\Delta\phi = 2\pi\left(\frac{1}{\sqrt{1-6w_c}} - 1\right) \approx 6\pi w_c$$

Now we must plug in numbers

- We must calculate w_c. W_c is a function of J, which is related to L, so we should compute it.
- Now L is defined by $$r^2\dot{\phi} = L$$
- Since the motion of Mercury is non-relativistic, we can replace λ derivative with time derivative.
- Since the orbit is almost circular, the velocity and radius are almost constant, therefore $$L = \frac{2\pi R^2}{T}$$

- R is the radius of the orbit, $R = 57{,}87 \times 10^9$ m
- T is the period, $T = 7{,}61 \times 10^6$ s
- Thus $L = 2.76 \times 10^{13}$ m^2/s
- To get J we must devide it by $2GM_{\oplus}/c$ and we finally get $J = 6250$
- Then $w_c = 6 / J^2$,

So finally

$$\Delta\phi \approx 6\pi w_c = 4.83\times10^{-7} / rev = 41.3''/100\,years$$