Relativistic Universe

General relativity IV:
Further developments

What we did …

- We introduced the geometry around massive body (point mass), and we derived equation for geodesics in this geometry.
- However some problems are still open, and now we will discuss them in turn.

Proper time

- We argued that there exists a very special choice of λ parameter, parametrizing the geodesics. What is its physical meaning?

The argument went as follows …

- Remember that

$$g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}=const$$

- By scaling $\lambda \rightarrow \lambda' = a\lambda$ we can make this constant equal +1, -1, or 0 (spacelike, timelike, or light-like.)

$$g_{\mu\nu}\frac{dx^{\mu}}{d\lambda'}\frac{dx^{\nu}}{d\lambda'}=a^{2}const$$

- In the timelike case $\lambda' = \tau$ is called proper time

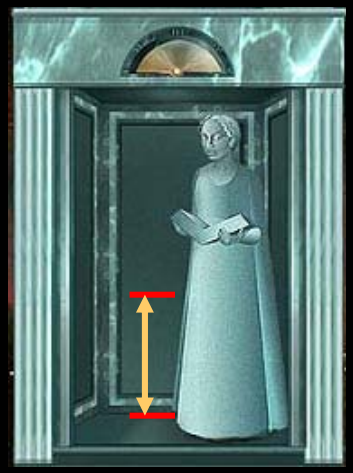
$$g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}=-1$$

What is the proper time?

- It follows that

$$d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu = -ds^2$$

- Now consider an observer in free fall. For him SR holds (so $g_{\mu\nu}=\eta_{\mu\nu}$)



- His clock measures the time dt

$$ds^2 = \eta_{00} dt^2 + 0 = -dt^2 = -d\tau^2$$

dx=0

Thus …

- The proper time is time as measured by free falling, locally Minkowski observer. Since the motion along geodesics is just the free fall, we can use τ as a natural parameter for the timelike ones with

$$g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}=-1$$

Conservation laws …

- In Newtonian mechanics laws of motion are second order in time derivatives.
- Sometimes, if symmetries are present, the order of equations can be reduced by one.
- Here we have similar situation, so …

What are the symmetries …

- Metric is ϕ independent; angular momentum conservation.

$$r^2 \frac{d\phi}{d\tau} = L = const$$

- Metric is t independent; energy conservation.

$$\left(1 - \frac{2m}{r}\right)\frac{dt}{d\tau} = E = const$$

The remaining equation

- Note that any body satisfies its own mass-shell condition.
- Let the components of momentum be

  $p_0 = -\mathcal{E}$, $p_\phi = \mathcal{L}$; then

Inverse metric

$$g_{\mu\nu} p^\mu p^\nu = g^{\mu\nu} p_\mu p_\nu = -\mu^2$$

Rest mass

Therefore

- Defining $E = \mathcal{E}/\mu$; $L = \mathcal{L}/\mu$ (energy and angular momentum per unit rest mass, after simple calculations we get

$$\left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{2m}{r}\right)\left(1 + \frac{L^2}{r^2}\right) = E^2$$

Local and asymptotic energy

- Consider an observer at rest, at distance *r* from the center. This observer measures energy of a particle moving in his vicinity to be E_{loc} (per unit mass).
- There is another observer at infinity, who measures energy of the same particle to be $E = const$ (this is the total energy of the particle.)

- It turns out that the relation between these quantities is

$$E = \sqrt{1 - \frac{2m}{r}}\, E_{loc}$$

Red shift

- To understand relevance of this formula, consider a photon emitted from a point at r towards infinity.
- At the point of emmision the photon has the wavelength $\lambda_{em}=h/E_{loc}$. At any point its wavelength satisfies

$$const = \sqrt{1-\frac{2m}{r}}\,E_{loc} = \frac{\lambda}{\sqrt{1-\frac{2m}{r}}}$$

- Therefore

$$\frac{\lambda_{em}}{\sqrt{1-\frac{2m}{r_{em}}}} = \frac{\lambda_{abs}}{\sqrt{1-\frac{2m}{r_{abs}}}}$$

- If $r_{em} = 2m$, then $\lambda_{abs} = \infty$

- In general we have red-shift for photon emitted at r and observed at infinity

$$z \equiv \frac{\lambda_{abs} - \lambda_{em}}{\lambda_{em}};$$

$$z = \frac{1}{\sqrt{1-\frac{2m}{r}}} - 1;$$

$$z = \frac{m}{r} \quad \text{in Newtonian limit}$$

Equation of motion

- Let us return to equation of motion

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2+\frac{1}{2}\left(1-\frac{2m}{r}\right)\left(1+\frac{L^2}{r^2}\right)=\frac{1}{2}E^2$$

- It describes motion in the potential

Newtonian

$$V(r)=\frac{1}{2}\kappa-\kappa\frac{m}{r}+\frac{L^2}{2r^2}-\frac{mL^2}{r^3}$$

κ=1 for timelike
κ=0 for lightlike

The extremum of potential …

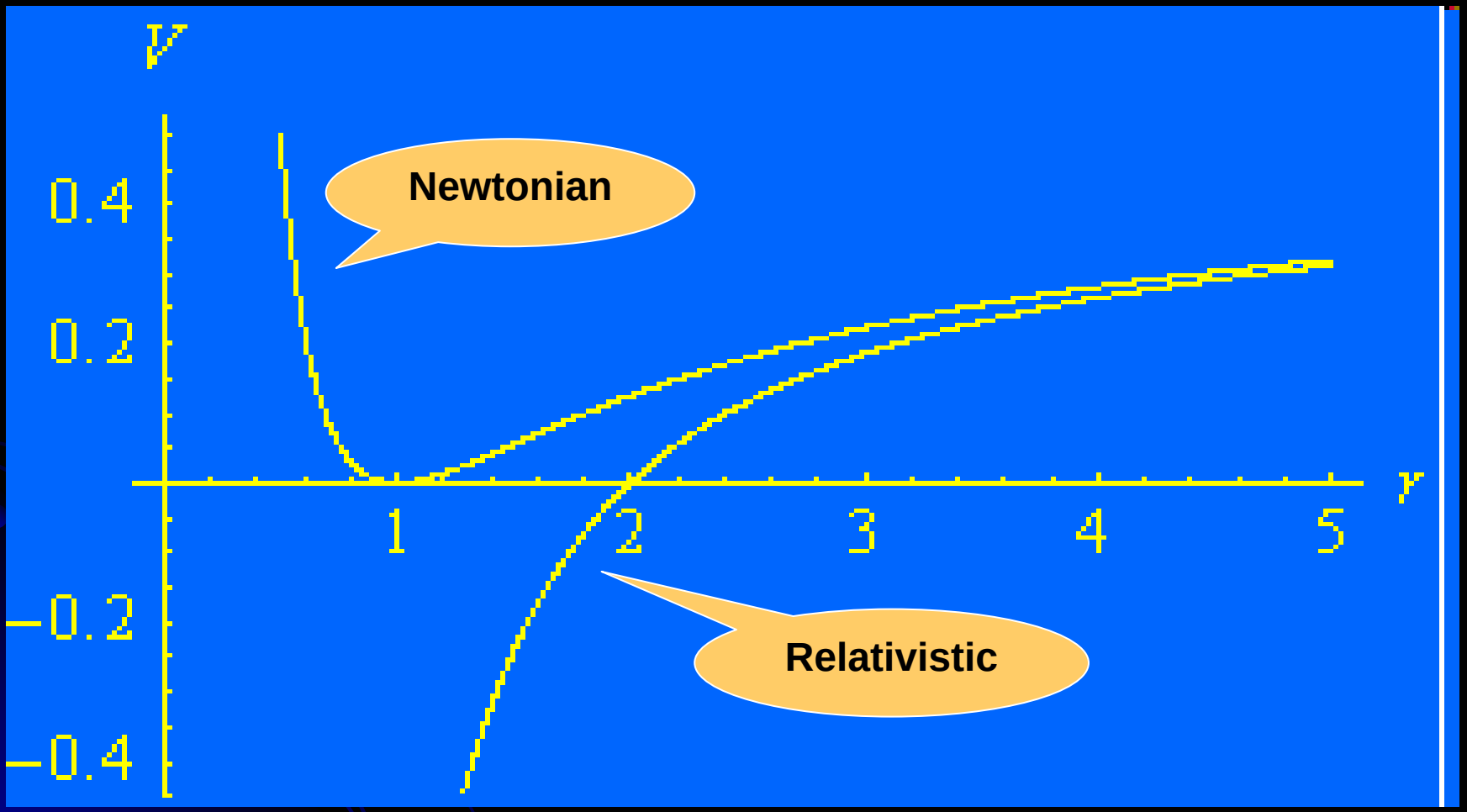
$$0 = \frac{dV}{dr} = \frac{mr^2 - L^2 r + 3mL^2}{r^4}$$

- has the roots

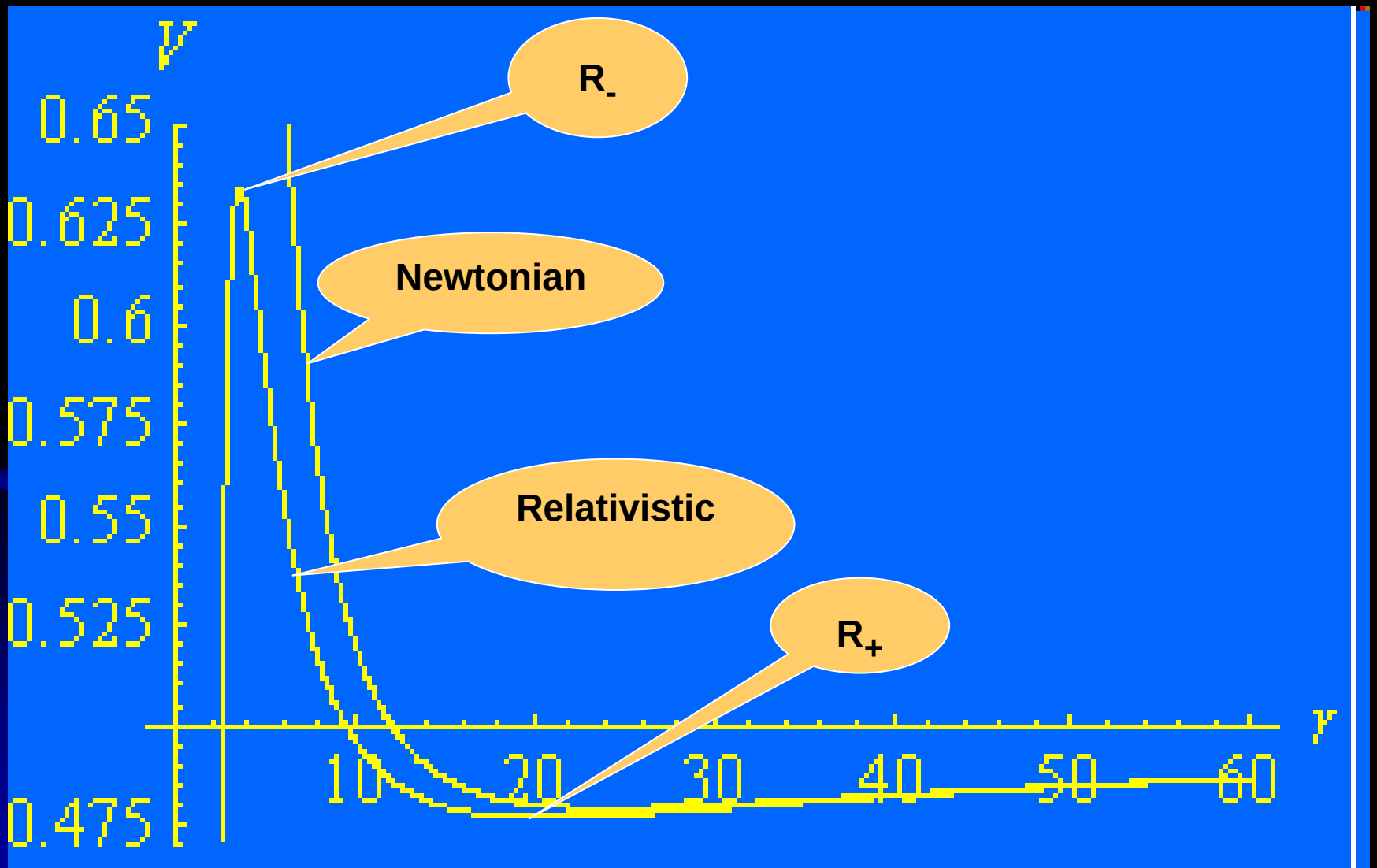
$$R_{\pm} = \frac{L^2 \pm \sqrt{L^4 - 12L^2 m^2}}{2m}$$

- No solutions for
  $L^2 < 12\, m^2$

The potential for $L^2 = m^2$



$L^2 = 24\ m^2,\ R_+ = 20,\ R_N = 24$



Orbits

- Stable circular orbit is possible if $R_+ > 6m$
- Unstable circular orbit is possible if for $3m < R_- < 6m$.
- A particle on circular orbit with $R >> m$ will slowly spiral to smaller radius, emitting gravitational radiation, and after reaching $R = 6m$ it will become unstable and fall rapidly towards $r = 0$.

Light orbits

- For light the potential is of the form

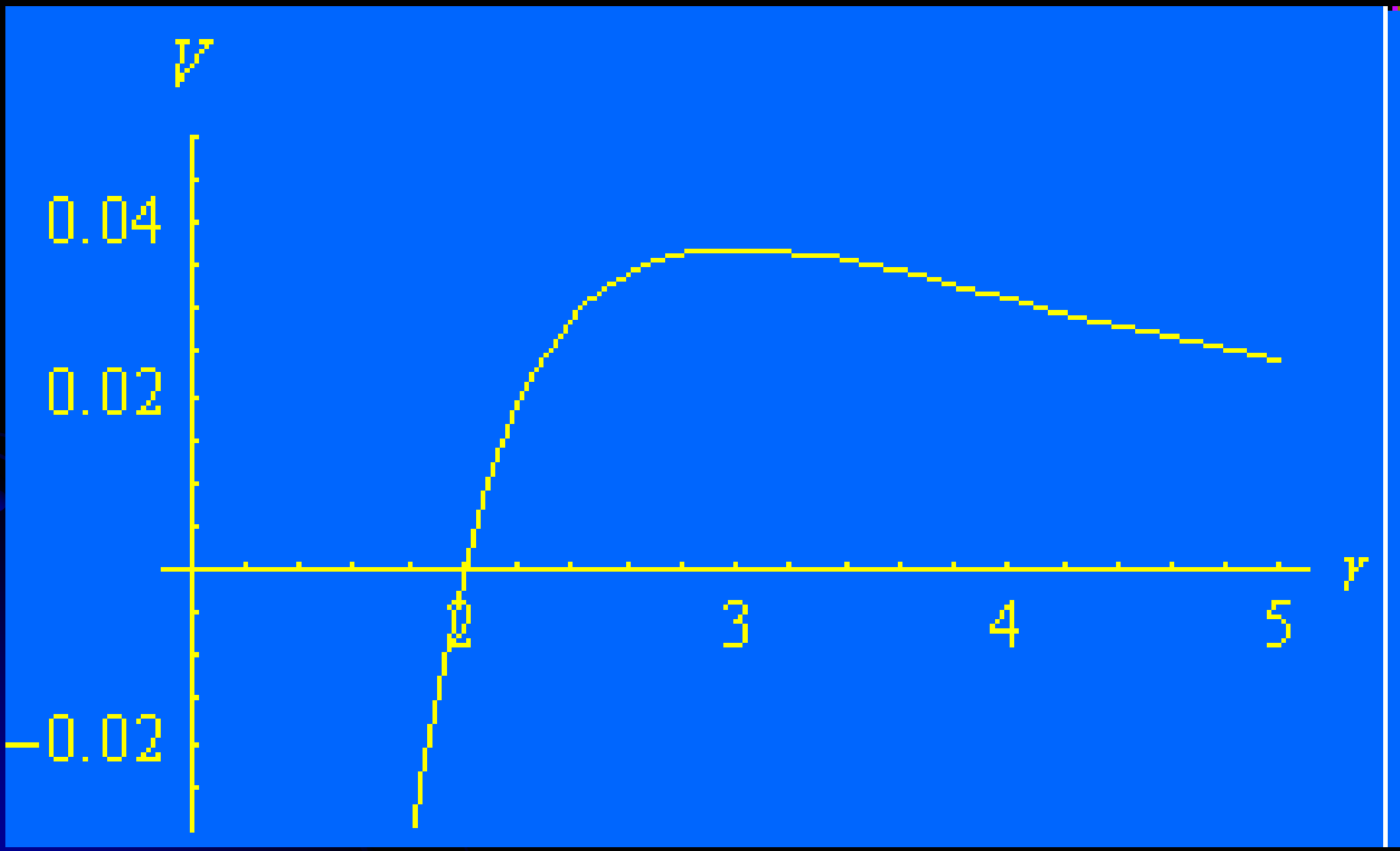
$$V = \frac{L^2}{2r^3}(r - 2m)$$

- This potential has single max at r=3m.

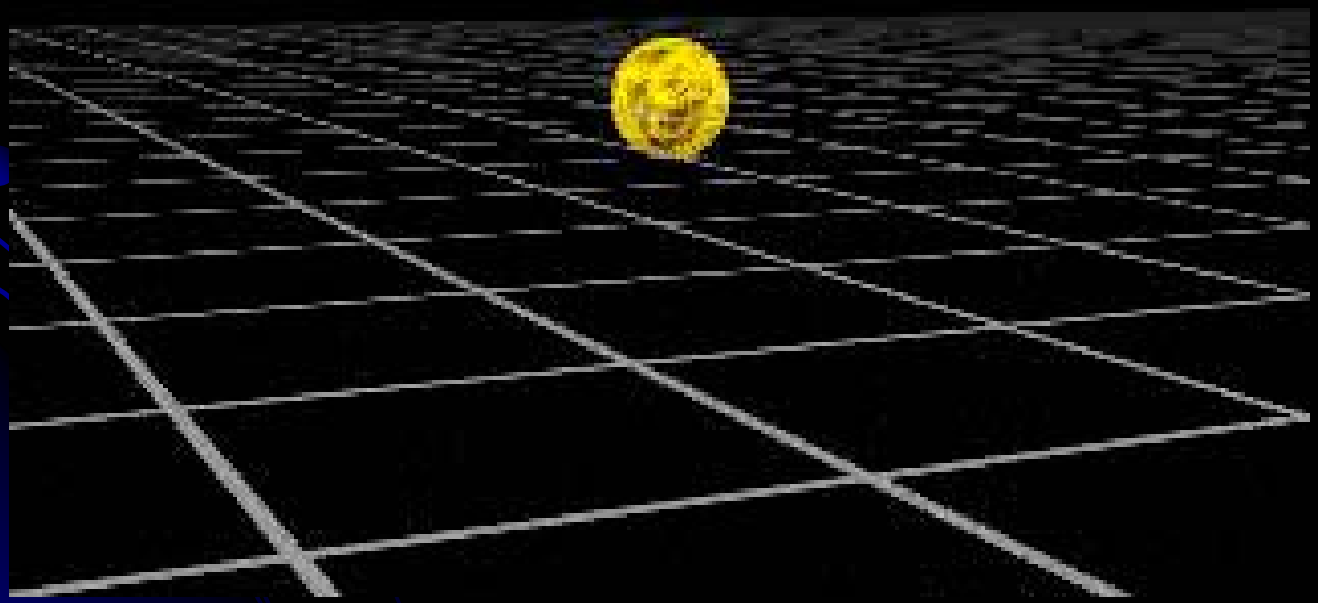
- At the maximum

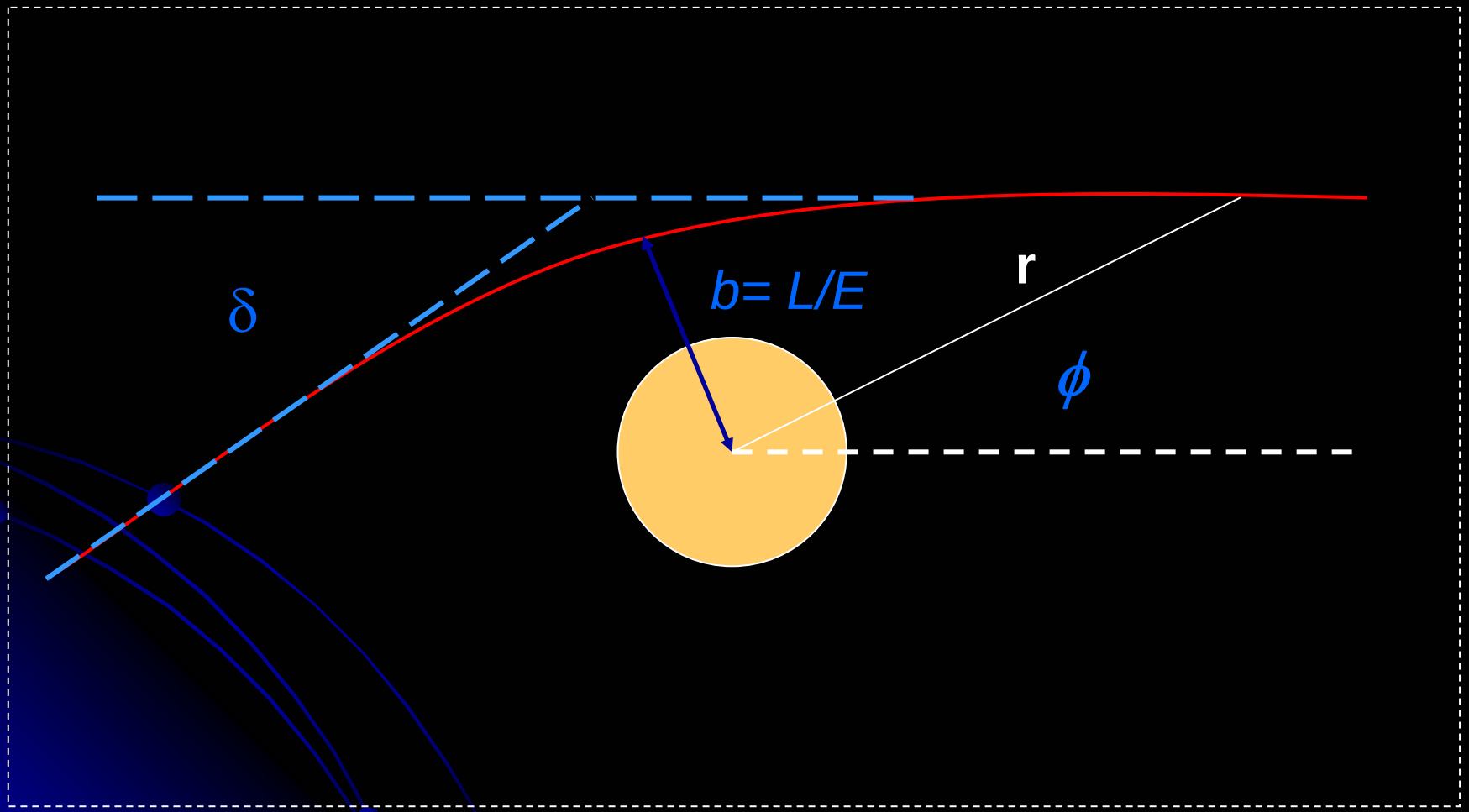
$$\frac{L^2}{E^2} = 27m^2$$

Potential for photons



Bending of light




δ
b= L/E
r
ϕ

Calculations

- The equation of motion is

$$\left(\frac{dw}{d\phi}\right)^2 + w^2 - 2w^3 = \frac{E^2}{J^2}$$

$$w = \frac{Gm}{r}, \quad J = \frac{L}{Gm}$$

- We can rewrite it as

$$\left(\frac{dw}{d\phi}\right)^2 + w^2 + \frac{E^2}{J^2} = 2w^3$$

oscillator

perturbation

First unperturbed equation

$$\left(\frac{dw_0}{d\phi}\right)^2 + w_0{}^2 + \frac{E^2}{J^2} = 0$$

$$w_0 = \frac{E}{J}\sin\phi$$

- In full equation one substirutes the solution w_0 to the RHS, and takes $w = w_0 + w_1$ with w_1 small

$$2\frac{E}{J}\left(\frac{dw_1}{d\phi}\right)\cos\phi + 2\frac{E}{J}w_1\sin\phi = 2\frac{E^3}{J^3}\sin^3\phi$$

$$w_1 = \frac{E^2}{J^2}\left(1+\cos^2\phi\right) + A\cos\phi$$

$$w(\phi) = \frac{E}{J}\sin\phi + \frac{E^2}{J^2}\left(1+\cos^2\phi\right) + A\cos\phi$$

The initial condition

- For the starting point $\phi = 0$ and $w(\phi) = 0$

$$\frac{1}{r}(\phi) = \frac{1}{b}\sin\phi + \frac{m}{b^2}\left(1+\cos^2\phi\right) - \frac{2m}{b^2}\cos\phi$$

Therefore

$$\frac{1}{r}(0) = 0$$

$$0 = \frac{1}{r}(\pi + \delta) = \frac{1}{b}\sin(\pi + \delta) + \frac{m}{b^2}\left(1 + \cos^2(\pi + \delta)\right) - \frac{2m}{b^2}\cos(\pi + \delta)$$

$$0 = -\frac{\delta}{b} + \frac{2m}{b^2} + \frac{2m}{b^2}$$

$$\delta = \frac{4Gm}{bc^2}$$

For Solar System experiment

$$\delta = \frac{4Gm_{\oplus}}{R_{\oplus}c^2} \approx 1.75''$$