Relativistic Universe

General relativity VI: Einstein equations and Friedman solution

Metric

 Locally any curved manifold is characterized by its metric (interval), which tells the distance between a point and points in its neighborhood.

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$$

Geodesic equation

 A test particle (the particle whose own gravitational field can be neglected) moves along geodesic

$$g_{\alpha\mu}\frac{d^{2}x^{\mu}}{d\lambda^{2}} + \frac{1}{2}\left(\frac{\partial g_{\alpha\mu}}{\partial x^{\nu}} + \frac{\partial g_{\alpha\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right)\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = 0$$

Christoffel symbols

 $\frac{d^2 x^{\alpha}}{d\lambda^2} + \Gamma^{\alpha}_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} = 0,$ $f_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\frac{\partial g_{\beta\mu}}{\partial x^{\nu}} + \frac{\partial g_{\beta\nu}}{\partial x^{\mu}} \right)$ $\frac{\partial g_{\mu\nu}}{\partial x^{\beta}}$ Γ^{α}

Covariant derivative

 In turning from flat to curved space one should replace partial derivatives with the covariant ones

$$\nabla_{\mu}v^{\alpha}(x) = \partial_{\mu}v^{\alpha}(x) + \Gamma^{\alpha}_{\mu\nu}(x)v^{\nu}(x)$$

$$\nabla_{\mu} v_{\alpha}(x) = \partial_{\mu} v_{\alpha}(x) - \Gamma^{\beta}_{\mu\alpha}(x) v_{\beta}(x)$$

 $\nabla_{\mu} t_{\alpha\beta}(x) = \partial_{\mu} t_{\alpha\beta}(x) - \Gamma^{\delta}_{\mu\alpha}(x) t_{\delta\beta}(x) - \Gamma^{\delta}_{\mu\beta}(x) t_{\alpha\delta}(x)$

Metric is covariantly constant

 $\nabla_{\mu}g_{\alpha\beta}(x) \equiv 0$

Riemann tensor

 Riemann tensor at a point measures curvature in the neighborhood of this point.

$$R^{\mu}_{\ \nu\alpha\beta} = \partial_{\alpha}\Gamma^{\mu}_{\ \nu\beta} - \partial_{\beta}\Gamma^{\mu}_{\ \nu\alpha} + \Gamma^{\mu}_{\ \sigma\alpha}\Gamma^{\sigma}_{\ \nu\beta} - \Gamma^{\mu}_{\ \sigma\beta}\Gamma^{\sigma}_{\ \nu\alpha}$$

Symmetries of Riemann tensor







Other curvature tensors

Ricci tensor

$$\begin{split} R_{\nu\beta} &= R^{\mu}_{\ \nu\mu\beta} \\ &= \partial_{\mu} \Gamma^{\mu}_{\ \nu\beta} - \partial_{\beta} \Gamma^{\mu}_{\ \nu\mu} + \Gamma^{\mu}_{\ \sigma\mu} \Gamma^{\sigma}_{\ \nu\beta} - \Gamma^{\mu}_{\ \sigma\beta} \Gamma^{\sigma}_{\ \nu\mu} \end{split}$$

Curvature scalar

$$R = g^{\nu\beta} R_{\nu\beta}$$

Example: the sphere

Metric

$$ds^{2} = r^{2} \left(d\theta^{2} + \sin^{2} \theta d\phi^{2} \right)$$
$$g_{\theta\theta} = r^{2}, \quad g_{\phi\phi} = r^{2} \sin^{2} \theta$$
$$g^{\theta\theta} = 1/r^{2}, \quad g^{\phi\phi} = 1/r^{2} \sin^{2} \theta$$

Christoffel symbols

$$\Gamma^{\theta}_{\ \phi\phi} = \frac{1}{2} g^{\theta\theta} \Big(2g_{\theta\phi,\phi} - g_{\phi\phi,\theta} \Big) = \frac{1}{2} \frac{1}{r^2} \Big(-2r^2 \sin\theta \cos\theta \Big) = -\sin\theta \cos\theta$$
$$\Gamma^{\phi}_{\ \theta\phi} = \Gamma^{\phi}_{\ \phi\theta} = \frac{1}{2} g^{\phi\phi} \Big(g_{\phi\theta,\phi} + g_{\phi\phi,\theta} - g_{\theta\phi,\phi} \Big) = \frac{1}{2} \frac{1}{r^2 \sin^2\theta} \Big(2r^2 \sin\theta \cos\theta \Big) = \frac{\cos\theta}{\sin\theta}$$

Riemann tensor

$$R_{\theta\phi\theta\phi}=r^2\sin^2\theta$$

• Ricci tensor

$$R_{\theta\theta} = \sin^2 \theta, \quad R_{\phi\phi} = 1$$

Curvature scalar

$$R = \frac{2}{r^2}$$

Bianchi identity

$$\nabla^{\mu} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \equiv 0$$

 This important identity is analogous to the identity for the LHS of the second pair of Maxwell equations

$$\varepsilon^{\mu\nu\rho\sigma}\partial_{\nu}F_{\rho\sigma}\equiv 0$$

Coupling to matter

- Gravity tells matter how to move
- Matter's energy and momentum tells spacetime how to curve
- In empty space Einstein equations have the form

$$R_{\mu\nu}=0$$

In the presence of matter the equation should be of the form

$$R_{\mu\nu} + \ldots = 8\pi G T_{\mu\nu}$$

• $T_{\mu\nu}$ is called energy-momentum tensor.

Properties of e-m tensor

- $T_{\mu\nu}$ is symmetric in μ , ν ;
- T_{00} is energy density, ρ ;
- T_{ii} are pressures in direction *i*, p_i ;
- For perfect isotropic, relativistic fluid, in the system, in which it is at rest

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

E-m tensor for electromagnetic field

$$\begin{split} T_{00} &= \frac{1}{8\pi} \Big(\vec{E}^2 + \vec{B}^2 \Big); \\ T_{0i} &= T_{i0} = -\frac{1}{4\pi} \Big(\vec{E} \times \vec{B} \Big)_i; \\ T_{jk} &= \frac{1}{4\pi} \Big(-(E_j E_k + B_j B_k) + \frac{1}{2} (\vec{E}^2 + \vec{B}^2) \delta_{jk} \Big) \end{split}$$

Equations of motion

• In flat space matter equations of motion can be expressed with the help of $T_{\mu\nu}$ as follows



For perfect fluid

 The general expression for energy momentum tensor of the fluid (gas) composed of particles is

$$T_{\mu\nu} = pg_{\mu\nu} + (\rho + p)u_{\mu}u_{\nu}$$



In flat space

- Componets of four velocity are u^μ=(1, νⁱ), where ν is much smaller than 1;
- Pressure is much smaller from energy density;

$$T^{00} = (\rho + p)u^{0}u^{0} + pg^{00} \approx \rho$$

$$T^{0j} = T^{j0} = (\rho + p)u^{0}u^{j} \approx \rho v^{j}$$

$$T^{jk} = (\rho + p)u^{j}u^{k} + p\delta^{jk} \approx \rho v^{j}v^{k} + p\delta^{jk}$$

Motion of the fluid

Continuity equation

$$0 = \partial_{\mu} T^{\mu 0} = \partial_{0} \rho + \partial_{i} \left(\rho v^{i} \right) = \dot{\rho} + \vec{\nabla} \left(\rho \vec{v} \right)$$

Euler equation

$$0 = \partial_0 T^{0j} + \partial_i T^{ij} = \dot{\vec{v}} + (\vec{v} \cdot \vec{\nabla})\vec{v} + \frac{1}{\rho}\vec{\nabla}p$$

Curved space generalization

 To generalize our result to curved space we must only replace partial derivative by the covariant one

$$\nabla_{\mu}T^{\mu\nu}=0$$

Einstein equations

• We know that

$$\nabla^{\mu} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \equiv 0$$

• Thus
$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

Consistency with vacuum eqns

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \implies$$
$$R - 2R = 0 \implies$$
$$R_{\mu\nu} = 0$$

Homogeneous and isotropic solution

- In many contexts we are interested in solution in which matter distribution is homogeneous and isotropic.
- In such a case the energy momentum tensor is the one of perfect fluid with energy density and pressure depending only on time.

Metric

 The homogeneous and isotropic metric can be brought to the form

$$ds^{2} = -dt^{2} + a^{2}(t) d\Omega^{(3)}$$

 dΩ⁽³⁾ is the metric of three-dimensional maximally symmetric space. There are only three such spaces

$$d\Omega^{(3)} = \frac{dr^2}{1 - r^2} + r^2 \left(d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \quad \text{3d sphere}$$

$$d\Omega^{(3)} = dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \quad \text{3d flat space}$$

$$d\Omega^{(3)} = \frac{dr^2}{1 + r^2} + r^2 \left(d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \quad \text{3d pseudosphere}$$

• It can be compactly rewritten as

$$d\Omega^{(3)} = \frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \quad k = +1, 0, -1$$

Einstein equation

• In the metric

$$ds^{2} = -dt^{2} + a^{2}(t) d\Omega^{(3)}$$

The only free parameter is *a(t)*

• The energy-momentum is

 $T_{\mu\nu} = p(t)g_{\mu\nu} + (\rho(t) + p(t))u_{\mu}u_{\nu}$

Einstein equations

• Friedman equations

$$\ddot{a} = -\frac{4\pi G}{3} (\rho + 3p)a$$
$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3}\rho$$

Integrability condition

 Differentiating second equation and eliminating second time derivative we find

$$\left(\rho a^3\right)^{\cdot} + 3pa^2\dot{a} = 0$$

Which is nothing but energy conservation