Relativistic Universe

Relativistic stars

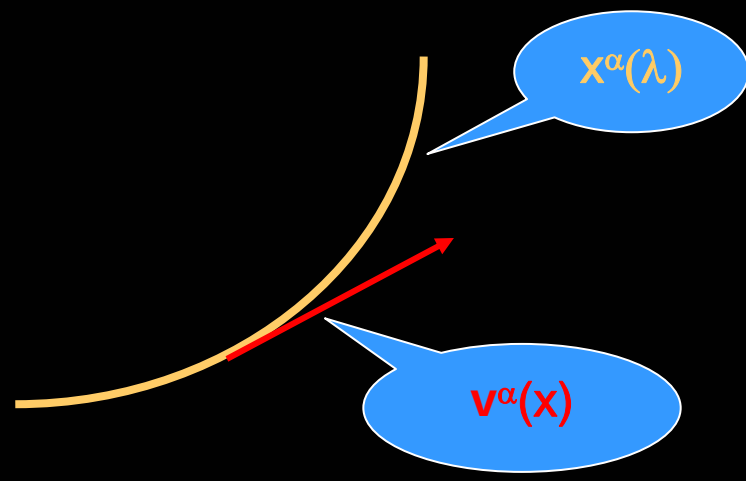
Metric

- Locally any curved manifold is characterized by its metric (interval), which tells the distance between a point and points in its neighborhood.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

- Tangent vector

$$v^{\alpha}(x) = \frac{dx^{\alpha}(\lambda)}{d\lambda}$$



- We can rewrite the geodesic equation as

$$\frac{d}{d\lambda} v^{\alpha}(x(\lambda)) + \Gamma^{\alpha}{}_{\mu\nu}(x) v^{\mu}(x) v^{\nu}(x) = 0$$

- But

$$\frac{d}{d\lambda} \equiv v^{\mu}\partial_{\mu}$$

- So that we can write the geodesic eqn. as

$$v^{\mu}\partial_{\mu}v^{\alpha}(x) + \Gamma^{\alpha}{}_{\mu\nu}(x)v^{\mu}(x)v^{\nu}(x) = 0,$$

$$0 = v^{\mu}\left(\partial_{\mu}v^{\alpha}(x) + \Gamma^{\alpha}{}_{\mu\nu}(x)v^{\nu}(x)\right) = v^{\mu}\left(\nabla_{\mu}v^{\alpha}(x)\right)$$

Covariant derivative

- The covariant derivative

$$\nabla_\mu v^\alpha(x) = \partial_\mu v^\alpha(x) + \Gamma^\alpha{}_{\mu\nu}(x) v^\nu(x)$$

- Is the right derivative, replacing partial derivative of the flat space and preserving tensor character of objects

Covariant derivatives of tensors

$$\nabla_\mu v^\alpha(x) = \partial_\mu v^\alpha(x) + \Gamma^\alpha{}_{\mu\nu}(x) v^\nu(x)$$

$$\nabla_\mu v_\alpha(x) = \partial_\mu v_\alpha(x) - \Gamma^\beta{}_{\mu\alpha}(x) v_\beta(x)$$

$$\nabla_\mu t_{\alpha\beta}(x) = \partial_\mu t_{\alpha\beta}(x) - \Gamma^\delta{}_{\mu\alpha}(x) t_{\delta\beta}(x) - \Gamma^\delta{}_{\mu\beta}(x) t_{\alpha\delta}(x)$$

It can be easily checked

- That the metric is covariantly constant

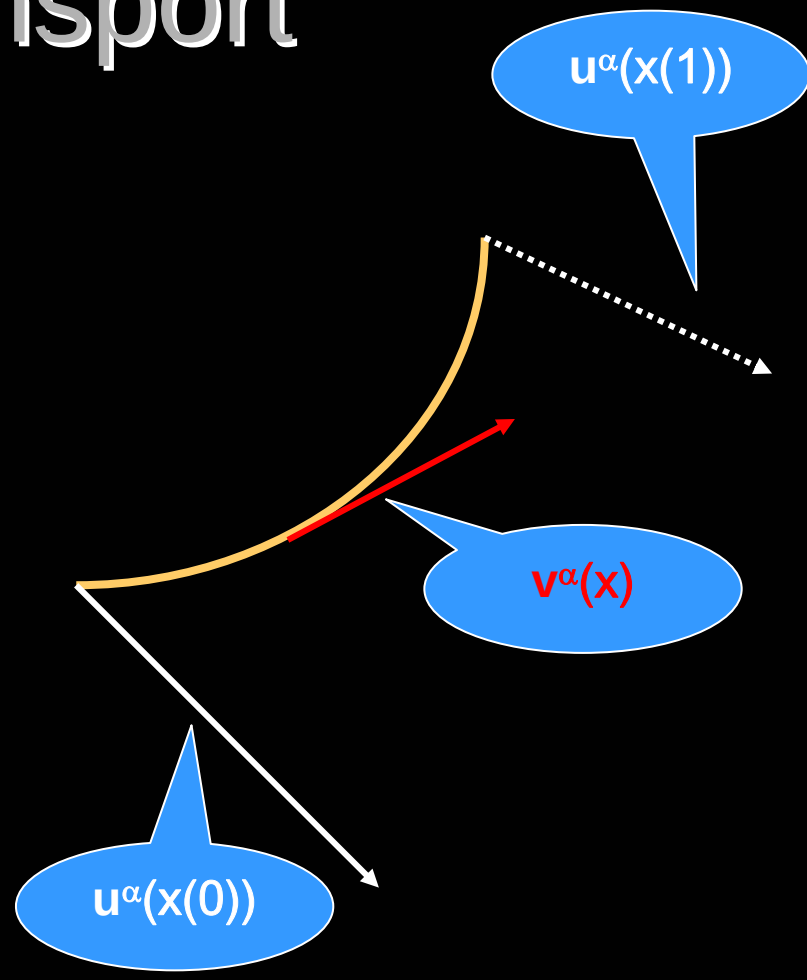
$$\nabla_{\mu} g_{\alpha\beta}(x) \equiv 0$$

Geometrical meaning

- In flat space, partial derivative ∂_μ is infinitesimal translation in direction μ.
- Similarly, in curved space, covariant derivative ∇_μ is infinitesimal parallel translation in direction μ.
- While turning to curved space we must replace all partial derivatives with covariant ones !

Parallel transport

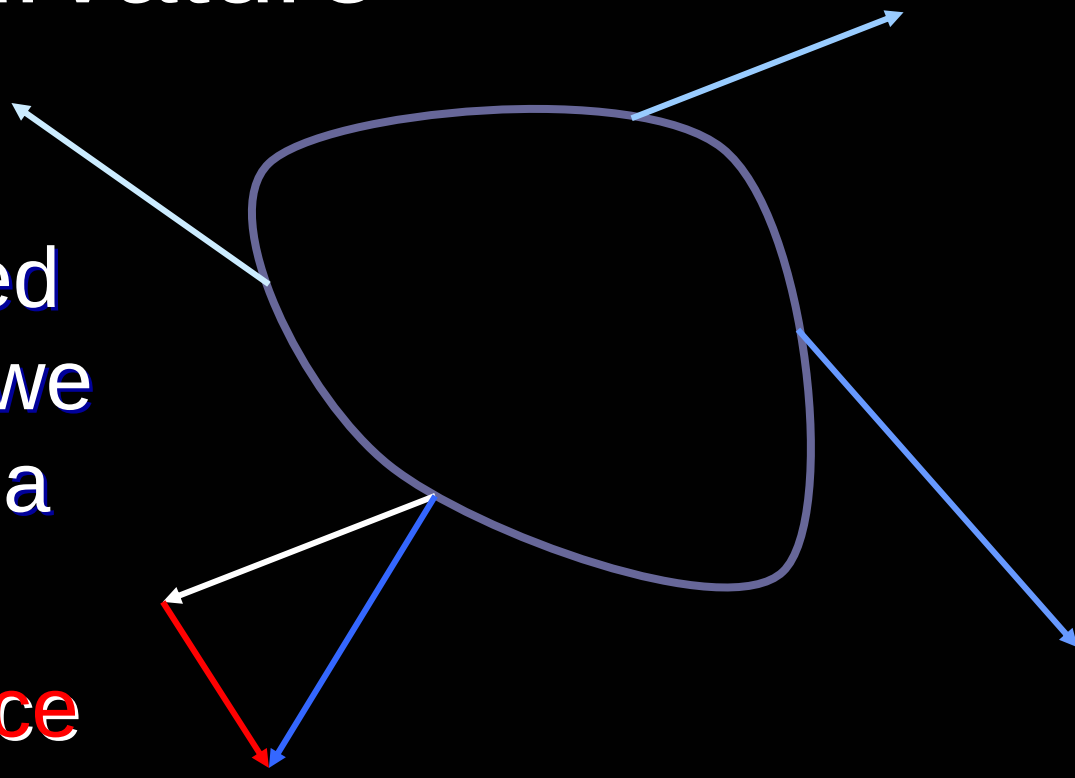
- If there is a curve with tangent vector $v^\alpha(x)$ and we want to paralel transport a vector $u^\alpha(x)$ along it we must solve the equation



$$v^\alpha \nabla_\alpha u^\beta = 0$$

Curvature

- If there is an infinitesimal closed curve (loop) and we parallel transport a vector along it
- Then the difference is proportional to curvature



Riemann tensor

- Curvature is measured by Riemann tensor, defined in terms of Christoffel symbols

$$R^{\mu}{}_{\nu\alpha\beta} = \partial_{\alpha}\Gamma^{\mu}{}_{\nu\beta} - \partial_{\beta}\Gamma^{\mu}{}_{\nu\alpha} + \Gamma^{\mu}{}_{\sigma\alpha}\Gamma^{\sigma}{}_{\nu\beta} - \Gamma^{\mu}{}_{\sigma\beta}\Gamma^{\sigma}{}_{\nu\alpha}$$

Symmetries of Riemann tensor

$$R_{\mu\nu\,\rho\sigma} = R_{\rho\sigma\,\mu\nu}$$

$$R_{\mu\nu\,\rho\sigma} = -R_{\nu\mu\,\rho\sigma}$$

$$R_{\mu\nu\,\rho\sigma} = -R_{\mu\nu\,\sigma\rho}$$

Contracting we get

- Ricci tensor

$$R_{\nu\beta} = R^{\mu}{}_{\nu\mu\beta}$$
$$= \partial_{\mu}\Gamma^{\mu}{}_{\nu\beta} - \partial_{\beta}\Gamma^{\mu}{}_{\nu\mu} + \Gamma^{\mu}{}_{\sigma\mu}\Gamma^{\sigma}{}_{\nu\beta} - \Gamma^{\mu}{}_{\sigma\beta}\Gamma^{\sigma}{}_{\nu\mu}$$

- Curvature scalar

$$R = g^{\nu\beta} R_{\nu\beta}$$

Vacuum Einstein equations

$$R_{\mu\nu} = 0$$

Bianchi identity

$$\nabla^{\mu}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)\equiv 0$$

- This important identity is analogous to the identity for the LHS of the second pair of Maxwell equations

$$\varepsilon^{\mu\nu\rho\sigma}\partial_{\nu}F_{\rho\sigma}\equiv 0$$

Coupling to matter

- Gravity tells matter how to move
- Matter's energy and momentum tells spacetime how to curve
- In empty space Einstein equations have the form

$$R_{\mu\nu} = 0$$

- In the presence of matter the equation should be of the form

$$R_{\mu\nu} + \ldots = 8\pi G\, T_{\mu\nu}$$

- $T_{\mu\nu}$ is called energy-momentum tensor.

Properties of e-m tensor

- $T_{\mu\nu}$ is symmetric in μ, ν;
- T_{00} is energy density, ρ;
- T_{ii} are pressures in direction i , p_i ;
- For perfect isotropic, relativistic fluid, in the system, in which it is at rest

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

Equations of motion

- In flat space matter equations of motion can be expressed with the help of $T_{\mu\nu}$ as follows

$$\partial^{\mu} T_{\mu\nu} = 0$$

For perfect fluid

- The general expression for energy momentum tensor of the fluid (gas) composed of particles is

$$T_{\mu\nu} = pg_{\mu\nu} + (\rho + p)u_{\mu}u_{\nu}$$

Four velocity

In flat space

- Componets of four velocity are $u^{\mu}=(1, v^{i})$, where v is much smaller than 1;
- Pressure is much smaller from energy density;

$$T^{00} = (\rho + p)u^0 u^0 + p g^{00} \approx \rho$$

$$T^{0j} = T^{j0} = (\rho + p)u^0 u^j \approx \rho v^j$$

$$T^{jk} = (\rho + p)u^j u^k + p\delta^{jk} \approx \rho v^j v^k + p\delta^{jk}$$

Motion of the fluid

- Continuity equation

$$0 = \partial_\mu T^{\mu 0} = \partial_0 \rho + \partial_i \left(\rho v^i \right) = \dot{\rho} + \vec{\nabla} \left(\rho \vec{v} \right)$$

- Euler equation

$$0 = \partial_0 T^{0j} + \partial_i T^{ij} = \dot{\vec{v}} + (\vec{v} \cdot \vec{\nabla}) \vec{v} + \frac{1}{\rho} \vec{\nabla} p$$

Curved space generalization

- To generalize our result to curved space we must only replace partial derivative by the covariant one

$$\nabla_{\mu} T^{\mu\nu} = 0$$

Einstein equations

- We know that

$$\nabla^{\mu}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)\equiv 0$$

- Thus (c=1)

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=8\pi G\,T_{\mu\nu}$$

Consistency with vacuum eqns

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \quad \Rightarrow$$

$$R - 2R = 0 \quad \Rightarrow$$

$$R_{\mu\nu} = 0$$

Metric of spherical relativistic star

- It turns out that the most general spherically symmetric metric is of the form

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2 d\Omega^2$$

- Λ(r) and Φ(r) both go to zero at infinity (far away from the source the metric is flat.)

Matter in star …

- Is described by

1. Energy density ρ (r),
2. Isotropic pressure p (r),
3. Barion number density n (r), with EOS;
4. 4-velocity of the fluid $u^{\mu} = u^{\mu}(r)$:

$$T^{\mu\nu} = (p + \rho)u^{\mu}u^{\nu} + pg^{\mu\nu}$$

00 Einstein equation

$$R_{00} - \frac{1}{2} g_{00} R = 8\pi G\, T_{00}$$

$$\frac{d}{dr}\left[r\left(1 - e^{-2\Lambda}\right)\right] = 8\pi G r^2 \rho$$

- Denoting (m(r) is mass in radius r)

$$r\left(1-e^{-2\Lambda}\right)=2Gm(r) \quad \Leftrightarrow e^{2\Lambda}=\left(1-\frac{2Gm(r)}{r}\right)^{-1}$$

- And solving the equation

$$\frac{2G}{r^2}\frac{dm(r)}{dr}=8\pi G\rho \Rightarrow$$

$$m(r)=\int_0^r 4\pi\rho dr$$

rr Einstein equation

$$R_{rr} - \frac{1}{2} g_{rr} R = 8\pi G \, T_{rr}$$

$$-\frac{1}{r^2}\left(1 - e^{-2\Lambda}\right) + \frac{2e^{-2\Lambda}}{r}\frac{d\Phi}{dr} = 8\pi G \; p$$

$$\frac{d\Phi}{dr} = G\frac{4\pi pr^3 + m}{r\left(r - 2Gm\right)}$$

- With Newtonian limit

$$\frac{d\Phi}{dr} = \frac{Gm}{r^2}$$

Oppenheimer – Volkov eqn.

- It follows from energy conservation that

$$\frac{dp}{dr} = -(\rho + p)\frac{d\Phi}{dr}$$

- So that we get

$$\frac{dp}{dr} = -G\frac{(p+\rho)(4\pi pr^3 + m)}{r\left(r - 2Gm\right)}$$

Nonrelativistic vs relativistic stars

	Nonrelativistic		Relativistic
(a)	$\frac{dM(r)}{dr} = 4\pi r^2 \rho$	Conservation of mass	$\frac{dM(r)}{dr} = 4\pi r^2 \rho$
(b)	$\frac{d\Omega}{dr} = \frac{GM(r)}{r^2}$	Hydrostatic equilibrium	$\frac{d\Omega}{dr} = \frac{G[M(r) + 4\pi r^3 P/c^2]}{r[r - 2GM(r)/c^2]}$
(c)	$\frac{dP}{dr} = -\frac{\rho\, d\Omega}{dr}$		$\frac{dP}{dr} = -\left(\rho + \frac{P}{c^2}\right)\frac{d\Omega}{dr}$
(d)	$\frac{dL(r)}{dr} = 4\pi r^2 \rho \epsilon$	Conservation of energy	$\epsilon = 0$
(e)	$P = \frac{\rho k T}{\mu m_h}$	Equation of state	$P = P(\rho)$
(f)	$\frac{dT}{dr} = f(P, T, \rho)$		The equation of state does not depend on T
(g)	$\kappa = \kappa(P, T, \rho)$		κ is irrelevant if $\epsilon = 0$

Solution of OV equation

- Assume that density of star is constant: $\rho(r) = \rho_0$.
- Then

$$m(r) = \int_0^r 4\pi\rho_0 dr \Rightarrow$$

$$m(r) = \frac{4\pi\rho_0}{3} r^3$$

- Then the OV equation takes the form

$$\frac{dp}{dr} = -\frac{4\pi G r \rho_0^2 (1 + p/\rho_0 c^2)(1 + 3p/\rho_0 c^2)}{3\left(1 - 8\pi G \rho_0 r^2 / 3c^2\right)}$$

- To which we add the usual initial condition p(R)=0

Solution

$$p(r) = \rho_0 c^2 \frac{\sqrt{1 - 2GMr^2 / R^3 c^2} - \sqrt{1 - 2GM / Rc^2}}{3\sqrt{1 - 2GM / Rc^2} - \sqrt{1 - 2GMr^2 / R^3 c^2}}$$

Central pressure

- At r=0

$$p(0) = p_c = \rho_0 c^2 \frac{1-\sqrt{1-2GM/Rc^2}}{3\sqrt{1-2GM/Rc^2}-1}$$

- This is the pressure in the center of a stable star of mass M and radius R

Condition of stability

- The central pressure becomes infinite if

$$R = \frac{9}{8}\frac{2GM}{c^2} = \frac{9}{8}R_s$$

- The star radius cannot be smaller if the star is to be stable.

- A more reasonable limit on the central pressure would be to limit the speed of sound to be less than or equal to the speed of light. A sound speed in excess of the speed of light would suggest conditions where the gas would violate the principle of causality. Namely, sound waves could propagate signals faster than the velocity of light. Since P/ρ_0 is the square of the local sound speed,

$$p_c = \rho_0 c^2 \frac{1-\sqrt{1-2GM/Rc^2}}{3\sqrt{1-2GM/Rc^2}-1} = \rho_0 c^2$$

$$R = \frac{4}{3}\frac{2GM}{c^2} = \frac{4}{3}R_s$$

Since causality will always dictate that the sound speed be less than the speed of light, we conclude that any stable star must have a radius R such that R > 4/3 R_s.

- From the NICMOS image we have seen that behind the obscuring dust, inside the Centaurus A Galaxy, there are a hot gas disk and a twisted jet and disk around a feeding black hole.

Matter falling on black hole

Black hole in globullar cluster