Relativistic Universe

General relativity V: Curvature and vacuum Einstein equations

Our goal is

- To find out what the dynamics of gravitational field is.
- We know that the gravitational field is associated with the metric, and according to the Second Newton's Law, its dynamics should be expressed in terms of the second order differential equation for the metric.

- Today we will restrict ourselves to the vacuum equations, i.e., gravity in absence of matter.
- But we know that metric tensor has a geometric meaning; it follows that the field equations for gravity should be expressed in terms of geometric objects as well.

Thus ...

• We must start with some differential geometry on curved manifolds.

A manifold ...

- Is some smooth hypersurface of dimension d (in our case d = 4) imbedded in R^N, N>d.
- Differential geometry deals with objects that are intrinsically defined on the manifold (independent of imbedding), like scalar functions, tangent vectors, tensors, etc.

Differentiation of functions

For flat space, if there is a scalar function *f(x)*, then its derivative along the vector with components *v*^α equals

$$\mathbf{v}(f)(x) = v^{\alpha} \frac{\partial}{\partial x^{\alpha}} f(x) = v^{\alpha} \partial_{\alpha} f(x)$$

 We assume that this definition holds in the curved case as well, independently of the coordinates

In particular

In Cartesian coordinates we have

$$\mathbf{v}(f)(x) = (v^x \partial_x + v^y \partial_y + v^z \partial_z) f(x, y, z)$$

and in spherical ones

$$\mathbf{v}(f)(x) = (v^r \partial_r + v^\theta \partial_\theta + v^\phi \partial_\phi) f(r, \theta, \phi)$$

Differentiation of vectors

 To differentiate we must move vector parallely along the curve and then compare



Affine connection

 Note that in order to differentiate vectors we need an additional structure (which is obvious in the flat space), telling how to parallel transport vectors from one point to another. This structure is called affine connection.

 We will not derive the way how to differentiate vectors, instead we use the geodesic equation to learn how this can be done

$$g_{\alpha\mu}\frac{d^{2}x^{\mu}}{d\lambda^{2}} + \frac{1}{2}\left(\frac{\partial g_{\alpha\mu}}{\partial x^{\nu}} + \frac{\partial g_{\alpha\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right)\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = 0$$

Can be rewritten as

$$\frac{d^{2}x^{\alpha}}{d\lambda^{2}} + \Gamma^{\alpha}{}_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = 0,$$

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(\frac{\partial g_{\beta\mu}}{\partial x^{\nu}} + \frac{\partial g_{\beta\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}}\right)$$

• $\Gamma^{\alpha}_{\mu\nu}$ are called Christoffel symbols

Tangent vector

$$v^{\alpha}(x) = \frac{dx^{\alpha}(\lambda)}{d\lambda}$$

• We can rewrite the geodesic equation as

$$\frac{d}{d\lambda}v^{\alpha}(x(\lambda)) + \Gamma^{\alpha}_{\mu\nu}(x)v^{\mu}(x)v^{\nu}(x) = 0$$



• But

$$\frac{d}{d\lambda} \equiv v^{\mu} \partial_{\mu}$$

• So that we can write the geodesic eqn. as

$$v^{\mu}\partial_{\mu}v^{\alpha}(x) + \Gamma^{\alpha}_{\mu\nu}(x)v^{\mu}(x)v^{\nu}(x) = 0,$$

$$0 = v^{\mu}\left(\partial_{\mu}v^{\alpha}(x) + \Gamma^{\alpha}_{\mu\nu}(x)v^{\nu}(x)\right) = v^{\mu}\left(\nabla_{\mu}v^{\alpha}(x)\right)$$

Covariant derivative

• The covariant derivative

$$\nabla_{\mu}v^{\alpha}(x) = \partial_{\mu}v^{\alpha}(x) + \Gamma^{\alpha}_{\mu\nu}(x)v^{\nu}(x)$$

 Is the right derivative, replacing partial derivative of the flat space and preserving tensor character of objects

Covariant derivatives of tensors

$$\nabla_{\mu}v^{\alpha}(x) = \partial_{\mu}v^{\alpha}(x) + \Gamma^{\alpha}_{\mu\nu}(x)v^{\nu}(x)$$

$$\downarrow$$

$$\nabla_{\mu}v_{\alpha}(x) = \partial_{\mu}v_{\alpha}(x) - \Gamma^{\beta}_{\mu\alpha}(x)v_{\beta}(x)$$

$$\nabla_{\mu} t_{\alpha\beta}(x) = \partial_{\mu} t_{\alpha\beta}(x) - \Gamma^{\delta}_{\mu\alpha}(x) t_{\delta\beta}(x) - \Gamma^{\delta}_{\mu\beta}(x) t_{\alpha\delta}(x)$$

It can be easily checked

That the metric is covariantly constant

$$\nabla_{\mu}g_{\alpha\beta}(x)\equiv 0$$



Geometrical meaning

- In flat space, partial derivative ∂_{μ} is infinitesimal translation in direction μ .
- Similarly, in curved space, covariant derivative ∇_μ is infinitesimal parallel translation in direction μ.
- While turning to curved space we must replace all partial derivatives with covariant ones !

Parallel transport

If there is a curve with tangent vector v^α(x) and we want to paralel transport a vector u^α(x) along it we must solve the equation





Curvature

If there is an infinitesimal closed curve (loop) and we parallel transport a vector along it Then the difference is proportional to curvature

Riemann tensor

 Curvature is measured by Riemann tensor, defined in terms of Christoffel symbols

$$R^{\mu}_{\ \nu\alpha\beta} = \partial_{\alpha}\Gamma^{\mu}_{\ \nu\beta} - \partial_{\beta}\Gamma^{\mu}_{\ \nu\alpha} + \Gamma^{\mu}_{\ \sigma\alpha}\Gamma^{\sigma}_{\ \nu\beta} - \Gamma^{\mu}_{\ \sigma\beta}\Gamma^{\sigma}_{\ \nu\alpha}$$

Contracting we get

Ricci tensor

$$\begin{split} R_{\nu\beta} &= R^{\mu}_{\ \nu\mu\beta} \\ &= \partial_{\mu} \Gamma^{\mu}_{\ \nu\beta} - \partial_{\beta} \Gamma^{\mu}_{\ \nu\mu} + \Gamma^{\mu}_{\ \sigma\mu} \Gamma^{\sigma}_{\ \nu\beta} - \Gamma^{\mu}_{\ \sigma\beta} \Gamma^{\sigma}_{\ \nu\mu} \end{split}$$

Curvature scalar

$$R = g^{\nu\beta} R_{\nu\beta}$$

Vacuum Einstein equations



Now let us check

- If the Schwarzschild metric is indeed a solution of vacuum Einstein equations.
- By doing that we will also convince ourselves that Einstein equations have right weak field limit, as the Schwarzschild metric does.

Recall that ...



The nonvanishing components of Riemann tensor are

$$R_{trtr} = \frac{2m}{r^3} = R_{rtrt},$$

$$R_{t\theta t\theta} = \frac{(2m-r)m}{r^2} = R_{\theta t\theta t},$$

$$R_{t\phi t\phi} = \frac{(2m-r)m\sin^2\theta}{r^2} = R_{\phi t\phi t},$$

$$R_{r\theta r\theta} = -\frac{m}{2m-r} = R_{\theta r\theta r},$$

$$R_{r\phi r\phi} = -\frac{m\sin^2\theta}{2m-r} = R_{\phi r\phi r},$$

$$R_{\theta \phi \theta \phi} = -2mr\sin^2\theta$$

So for example

$$\begin{aligned} R_{tt} &= g^{rr} R_{rtrt} + g^{\theta \theta} R_{\theta t \theta t} + g^{\phi \phi} R_{\phi t \phi t} = \\ \frac{r - 2m}{r} \frac{2m}{r^3} + \frac{1}{r^2} \frac{(2m - r)m}{r^2} + \frac{1}{r^2 \sin^2 \theta} \frac{(2m - r)m \sin^2 \theta}{r^2} = 0 \end{aligned}$$